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Heterogeneous Agent Models of Simple Exchange Economies:

on the Role of Investment Horizons and Wealth Dynamics

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Abstract

The thesis is devoted to the analytical modeling of simple exchange economies in the presence of heterogeneous, boundedly rational traders. We present a range of simple models where the consequence of Walrasian equilibrium can be described by dynamical systems.

In the model presented in the first part of the thesis we explore the effects of the differences in the investment horizons on the price dynamics in market with two types of investors: fundamentalists and chartists. Considering the model without chartists, we find that the sole heterogeneity with respect to the time horizons is not enough to bring instability to the system and generate a non-trivial price dynamics. However, if agents of both types are present in the market, the heterogeneity of investment horizons can enlarge the parametric region of instability. One of the reasons behind this phenomenon is that the stability of the system is, ultimately, related to the *relative demand* of different traders' types.

The model in the second part of the thesis deals with the market where the dynamics is driven by an interweaving between evolutions of return and wealth. The underlying idea of such approach is that more wealthy agents have greater impact on the price determination than less wealthy ones. Consequently, traders demand for the risky asset is expressed as a fraction of their individual wealth and is based on future price forecast obtained on the basis of past market history. In the most general version of the model an arbitrary large number of heterogeneous traders operate in the market and any smooth function which maps the infinite information set to the present investment choice is allowed as agent's trading behavior. The set of equilibria of the corresponding dynamical system(s) are studied and their stability conditions are derived. It is shown that abstracting from precise specification of the agents' investment decisions, all possible long-run equilibria belong to a one dimensional *Equilibrium Market Line* (EML). The relative performances of different strategies are discussed and selection principle with which market endogenously selects the dominant behaviors is formulated. The results point to the simple explanation of some findings in the recent analytical and simulation contributions.

Chapter 1

Introduction

This thesis presents a range of simple dynamical models of speculative markets where boundedly rational, heterogeneous agents trade in a Walrasian equilibrium framework.

The main motivation is the impossibility to explain within the standard paradigm of the Efficient Market Hypothesis (EMH) (see Fama (1970) for a review) most of the stylized facts characterizing the dynamics of financial markets. According to the EMH the asset prices are completely determined by economic fundamentals. Shiller (1981) demonstrated that the price volatility implied by such statement is substantially lower than the volatility observed in real markets. West (1988), summarizing the results of several econometric investigations, concludes that the phenomenon of *excess volatility* cannot be rejected. Relatedly, even if the EMH implies that movements in the asset prices are related with the change in fundamental conditions, Frankel and Froot (1986) present the historical evidence that the 1981-1984 appreciation of dollar cannot be explained through the changes in the fundamentals. The evidence of persistent deviations of price from its fundamental evaluations was also confirmed for the stock markets in Poterba and Summers (1988). Another famous stylized fact, *volatility clustering*, says that large changes in prices tend to be followed by large changes, while small changes tend to be followed by small changes. Therefore, the absolute changes in asset prices are predictable (while returns themselves are not). After the seminal paper Engle (1982) a number of ARCH-type models have been developed in order to estimate (but not to explain) this phenomenon. Finally, standard asset pricing models would predict the absence of trade in the market, whereas the daily trading volume in real markets is huge. For a detailed review of the stylized facts, see Brock (1997) and Guillaume, Dacorogna, Davé, Müller, Olsen, and Pictet (1997).

Another problem of standard financial models is the set of unrealistic assumptions on which they are based. The famous Milton Friedman's essay (see Friedman (1953)) made popular the so-called "as if" principle defending the use of possibly unrealistic assumptions and proclaiming that

> the question whether a theory is realistic "enough" can be settled only by seeing whether it yields predictions that are good enough for the purpose in hand or that are better than predictions from alternative theories. (p.41)

Despite the intensive debate pointing to the fallacy of this logic[1], the models based upon principles of individual rationality, unlimited capability in avoiding forecasting mistakes, homogeneity of agents' behavior and equilibrium nature of the markets became common in economics, in general, and in finance, in particular. For example, the text-book version of

[1]See, for example, Samuelson (1963), Simon (1963) and Simon (1979).

the Capital Asset Pricing Model (CAPM) introduced in Sharpe (1964) and Lintner (1965) assumes both individual rationality and homogeneity of expectations. The homogeneity in preferences is not necessary but usually assumed in the process of econometric testing. The same applies to the dynamical generalization of CAPM in Merton (1973). The solution of the latter model can be, indeed, found only under quite restrictive requirements. Ross (1978) points out that these requirements (in particular, the logarithmic utility maximization) leads to a contradiction, because the implied equilibrium price behavior would become deterministic. As a solution of the problem he proposes the "rational price functions" introduced in LeRoy (1973). Such *rational expectations* approach was formalized in the pure exchange economy model of Lucas (1978).

All these assumptions have been extensively criticized both on empirical and theoretical grounds. The hypothesis of behavioral homogeneity is in contradiction with the presence of technical trading activity in the markets and also with the disagreement of investors about fundamentals. See, e.g. studies in Frankel and Froot (1987) and Allen and Taylor (1990), which are based upon the direct surveying of individual investors. Possible escape from the heterogeneity problem is through the introduction of a so-called *representative agent* who exhibits "average" agent's behavior and, allows to reduce the investigation of the aggregate economic properties to the homogeneous case. This approach has been criticized at length in Kirman (1992).

Theoretical arguments against an assumption of individual rationality were developed by Herbert Simon and led to the development of the *bounded rationality* approach. Experimental economics supports bounded rationality providing an extensive evidence against individual rationality (see, e.g. Kahneman and Tversky (1979) and Tversky and Kahneman (1986)). Along these lines, Sargent (1993) suggests to model individual agents as actors who do not possess compete knowledge about the economic system and, therefore, have to be satisfied with econometric analysis of the observed series. Similarly, Arthur (1994) proposes the methodology of inductive reasoning when the agents apply relatively simple predictors or forecasting rules and learn from the past about performances of these predictors.

The enormous discrepancy between real markets and its models, both in predictions and assumptions, has been the driving force behind the emergence, in relatively recent times, of a new strand of models, generally referred to as *agent-based models*, based upon the idea that markets can be described as complex systems of interacting *boundedly rational* and *heterogeneous* agents. The analysis of these models has focused both on the emerging properties of the generated time series of prices, returns and volumes and also on the role the ecology of different agents' behavior plays in the emergence of these properties. See LeBaron (2000) for a review of early investigations and the discussion about the origins of the agent-based approach.

Inside the large and rapidly growing body of contributions to the agent-based literature on financial markets, one can roughly recognize two groups of models. The first group contains models essentially built to be simulated on computers. Among the vast literature, see contributions of Levy, Levy, and Solomon (1994), Arthur, Holland, LeBaron, Palmer, and Tayler (1997), LeBaron, Arthur, and Palmer (1999), Lux and Marchesi (1999), Kirman and Teyssire (2002), Farmer (2002) and Bottazzi, Dosi, and Rebesco (2005). The analysis in the simulation models is concentrated on the statistical properties of generated time series, usually confronted with stylized facts. Recent survey in LeBaron (2005) contains up-to-date overview of the achievements of this approach. Levy, Levy, and Solomon (2000) explains the interest in the simulation modeling by the absence of *any* restrictions on the side of assumptions. Simulation models indeed allow high degree of flexibility to the researcher. On the other hand, the systematic study of such models is made practically impossible by the enormous number of degrees of freedom. As a result, it is not always clear which assumptions are crucial for generated patterns.

The second group is composed by models explicitly built to be analytically tractable. These models have the advantage that their dynamics can be studied using the powerful tools of mathematical analysis. Consequently, one can investigate different theoretical questions. For instance, model in Beja and Goldman (1980) demonstrates that the transactions which happen due to the finite speed of price adjustment towards the equilibrium value can destabilize the market. DeLong, Shleifer, Summers, and Waldmann (1990a) build a model where the instability is generated, inside a heterogeneous environment, by (otherwise stabilizing) speculations of fundamental traders. DeLong, Shleifer, Summers, and Waldmann (1991) showed that irrational traders can survive in the market competition against rational agents. Brock and Hommes (1997) demonstrate that different forecasting rules can survive an evolutionary selection based upon their performances and that corresponding dynamics is highly non-linear and even chaotic. Examples of analytical models can be found in the contributions by Frankel and Froot (1990), Day and Huang (1990), DeLong, Shleifer, Summers, and Waldmann (1990b), Topol (1991), Chiarella (1992) Kirman (1993), Lux (1995), Lux (1997), Lux (1998), Brock and Hommes (1998), Gaunersdorfer (2000), Hommes (2001) Chiarella and He (2001), Chiarella and He (2002b), Chiarella and He (2002a), Chiarella and He (2003), Brock, Hommes, and Wagener (2005). The achievements of this branch of the literature are summarized in the recent review of Hommes (2005). Following this review, the analytical models of financial market are referred as *heterogeneous agent models* in this thesis.

It is also important to mention that quite simple deterministic models with a small degree of heterogeneity can generate time series of price with properties closed to those of real data and also phenomena similar to observed "excess volatility" or "volatility clustering". See the discussion about possibility to explain these stylized facts through the property of nonlinear deterministic systems in Gaunersdorfer, Hommes, and Wagener (2003) and Gaunersdorfer and Hommes (2005).

This thesis contributes to the analytical branch of the agent-based models. In the first part, the model where agents are heterogeneous in the investment horizons is proposed and analyzed. The second part is devoted to the development of the analytical model of the market in which both price and wealth are endogenously evolving in the intertwined fashion.

Below the short review of the following Chapters is provided. Model in Chapter 2 is based upon Anufriev and Bottazzi (2004), the model in Chapter 4 represents a generalized version of Anufriev, Bottazzi, and Pancotto (2004), and, finally, the model in Chapter 5 closely follows Anufriev and Bottazzi (2005).

Thesis Structure

Chapter 2

In the next Chapter we study the influence of the investors' time horizons on the market dynamics in a simple asset pricing model. The economy in the model contains two assets: a bond with risk-less return and a stock paying a constant dividend. The price of the stock is determined through a market clearing condition. Traders are speculators with different and fixed investment horizons. Their demands are derived as the solutions of the mean-variance maximization problem over proper horizons. Agents also differ in the perceptions about future price fluctuations, so that two traders' types, representing typical trading attitudes, are considered, fundamentalists and chartists. Traders in the model are boundedly rational in the sense that their comprehension of the next period expected return and its variance is based upon past market performance. As a consequence, even if fundamental conditions are known to everybody with certainty, the market can display, under suitable parameterization, a cyclical or chaotic price dynamics with speculative bubbles and crashes.

We analyze in this model the effect of the differences in investment horizons on the price dynamics. Considering the model without chartists, we find that the sole heterogeneity with respect to the time horizons is not enough to bring instability to the system and generate a non-trivial price dynamics. However, if agents of both types are present in the market, the heterogeneity of investment horizons can enlarge the parametric region of instability. We argue that the ultimate reason for this phenomenon lies in the fact that the stability/instability of the system is related with the *relative demand* of different traders' types. For example, if the investment horizon of fundamentalists increases but they overestimate the long-term risk, then the relative demand of fundamentalists decreases and the system may loose its stability.

Chapter 3

This and the following four Chapters are devoted to the model where endogenous return dynamics is accompanied by an endogenous and interrelated wealth dynamics. The underlying idea of such approach is that more wealthy agents have greater impact on the price determination than less wealthy. In this Chapter we formally develop the framework which incorporates such effect and review the related literature.

We propose a discrete-time model of pure exchange economy with two assets, one riskless, yielding a constant return on investment, and one risky, paying a stochastic dividend. One important assumption concerns the agents' demand. Namely, we assume that it is expressed as a fraction of agent's wealth, which, in general, depends on the past return history through a smooth *investment function*. The number of different investment functions, i.e. agents' types, is in this model arbitrary. The price of the risky asset is fixed by imposing a market clearing condition on the sum of the traders' individual demand functions. In such framework the asset's price and agents' wealth are determined at the same time and the resulting market evolution is described by a multi-dimensional system of difference equations. The main innovation of our approach with respect to the previous contributions is to leave the agent-specific investment functions unspecified. The resulting system is analyzed in the following Chapters.

Chapter 4

The system introduced in Chapter 3 is analyzed here under the assumption that agents' investment functions depend on the forecast about future return. This forecast is obtained through the exponential weighted moving average (EWMA) estimators with agent-specific weights.

We study the set of equilibria of this system and provide an asymptotic characterization of the agents' relative performances. We derive the necessary and sufficient conditions under which the system possesses "no-arbitrage" equilibria where both assets are equivalent. Apart from these equilibria, the economy can only possess isolated generic equilibria where a single agent dominates the market and continuous manifolds of non-generic equilibria where many agents hold finite wealth shares. In both cases the economy's growth rate is determined by the growth rate of surviving agents. Abstracting from precise specification of the agents' investment decisions, we show that all possible long-run equilibria belong to a one dimensional *Equilibrium Market Line* (EML).

The stability conditions of equilibria are derived. The results point to the simple explanation of some findings in the recent simulation paper by Zschischang and Lux (2001) and, in general, can be applied to the investigation of the role of different parameters for the stability of the market dynamics.

Chapter 5

In this Chapter we consider another specification of the model from Chapter 3. The general case is studied in which an arbitrary large number of heterogeneous traders operate in the market and any smooth function which maps the infinite information set to the present investment choice is allowed as agent's trading behavior. A complete characterization of equilibria is given and their stability conditions are derived. As in Chapter 4, one dimensional Equilibrium Market Line can be applied for the illustration of possible equilibria of the system.

The stability conditions lead us to the discussion of the general, *selection principle* with which market endogenously selects the dominant traders. We demonstrate that this principle displays some degree of optimality in the neighborhood of equilibria, but, at the same time, leads to the impossibility to define a global dominance order relation among strategies.

Chapter 6

This Chapter aims to demonstrate how the geometric illustration derived from the rigorous quantitative machinery developed in the three previous Chapters can be applied to the study of models with a particular subset of investment strategies. We demonstrate two results here. First, we accurately prove that all the finding of the contribution Chiarella and He (2001) can be easily re-obtained inside our framework. The use of the Equilibrium Market Line allows us to obtain all the results in a more simple and elegant way and also to reveal the limitations of previous investigations. Second, we show that rational behavior (in the sense of mean-variance optimization) leads to peculiar "linear" investment functions. The analysis of these functions highlights those features of the asymptotic market dynamics which are common to all types of allowed investment behavior and those which are specific for these "linear" strategies. In particular, we find that in the market with such strategies, only particular types of equilibria can be observed.

Chapter 7

Here we analyze the stochastic versions of the model introduced in Chapter 3. We, first, consider the effects of the random yield for the dynamics of the market and justify the analysis of the deterministic version of the system performed in Chapters 4 and 5. Second, we analyze to what extent and how the trading activity of a group of heterogeneous agents can be described, in the aggregate, as the result of the investment decision of a single "representative" agent. It is shown that if individual choices are expressed as noisy versions of a common behavior, and the number of agents is large, one can consider the *Large Market Limit* of the economy and reduce the model to a low-dimensional stochastic system. The goodness of this approximation under different market conditions and different agents ecologies is analyzed through numerical simulations.

Part I

Model with Heterogeneous Investment Horizons

Chapter 2

Asset Pricing Model with Heterogeneous Investment Horizons

In this Chapter we present a simple dynamic model of speculative markets where mean-variance maximizing, boundedly rational agents trade in a Walrasian equilibrium framework. The present work extends the model in Bottazzi (2002) introducing heterogeneity with respect to the time horizons on which agents base their investment decisions and, consequently, their trading activity[1].

2.1 Introduction

The main motivation for the current investigation lies in the fact that the heterogeneous agent models usually assume that the demand of each investor depends on his expectations for the next period. In other words, the agents are "myopic" in their decisions. Such myopia implies the homogeneity of the time horizons of different agents, and also that all investors have the same frequency of their operations in the financial market. This simplifying assumption is made despite the opposite evidence from the real markets[2]. The question which arises is whether the predictions of that models, like possible instability of the price time series due to heterogeneity in expectations, are valid in the presence of the traders with different investment horizons. In more general terms, one would like to know the possible effects which investors' time horizons and their heterogeneity bring to the market. As we will shortly discuss below, in the classical models with exogenous price determination such question naturally emerged in the end of 60's. Nowadays, when the models with endogenous pricing attract greater attention, it

[1]We use in this Chapter the term "investment horizon" and, equivalently, "time horizon" in the sense in which financial institutions do. That is, we refer to the length of time for which an agent would like to keep his investments in order to get the return afterwards. "Investment horizon" with the same meaning was also used in the classical economic literature, e.g. in the recent monography Campbell and Viceira (2002). Osler (1995) uses "speculators' horizon" for the same concept. On the other hand, some scholars used the term "horizon" with a different meaning. In LeBaron (2001) "time horizon" stands for the number of periods in the *past* which are taken into consideration by technical traders in choosing their investment strategy. To avoid any misunderstanding, we want to stress here that these two concepts of "horizons" are completely different and unrelated. In our paper, the "horizon" extends in the future, while in LeBaron (2001) "horizon" is something referring to the past.

[2]See, e.g. Dacorogna, Müller, Jost, Pictet, Olsen, and Ward (1995) who say that "the variety of time horizons is large: from intra-day dealers, who close their positions every evening, to long-term investors and central banks"(p. 384)

is interesting and important to enlarge the knowledge about impact of the investment horizons on the market.

Our model contributes also to the question about minimal heterogeneity requirements which are enough to generate such phenomena as excess volatility or volatility clustering. It is commonly believed that heterogeneity of traders in the market plays the main role in the explanation of these "stylized facts". The question then whether the heterogeneity in expectations is necessary for generating these phenomena or other sources of heterogeneity (e.g. attitude towards the risk, investment horizons, available information) can also be sufficient. This model supports the view that the heterogeneity in expectations is the most important. At the same time, this model stresses that expectations, and, therefore, the qualitative aspects of the dynamics are affected by the horizons.

The structure of the model is kept extremely simple. We consider an economy with two assets: one risk-less bond and one risky equity whose price is determined via a Walrasian auction. The market participants are described as mean-variance utility maximizers. They choose their portfolio composition at each period based on their forecast of the future return. Even if the model allows a high level of heterogeneity among traders, we restrict our analysis to the case in which agents are taken as representatives of two distinct classes intended to stylize two basic attitudes toward market participation. The first class, *fundamentalists*, represents traders who obtain future return predictions on the base of the fundamental value of the asset. The second class, *chartists*, represents traders whose forecasts are based on the past return history.

Even if the basic set-up of the model shares the same spirit as some previous contributions (Brock and Hommes, 1998; Brock, Hommes, and Wagener, 2005), it differs from them in some important aspects. First, in our model investors form expectations about return and not about price. This assumption is more convenient but also more realistic as we discuss below. Second, we consider traders who explicitly take into account not only expected return but also the *risk* involved in their market positions, which is rather natural in the mean-variance framework. Since under these two assumptions the model is capable to generate complex, e.g. chaotic, dynamics, we also assume that the relative shares of different trading behaviors (chartists and fundamentalists) are fixed. It stresses that the main players in the financial markets are led by their specific strategies whose frequency of updating is much less than the frequency of generated price fluctuations. Accordingly, the investment horizons are assumed to be fixed for two traders' classes.

The analysis of the present model reveals two interesting aspects. On the one hand, we find that in the market without chartists, the dynamics is stable even if horizons of fundamentalists are different. Therefore, the heterogeneity in time horizons alone is not enough to generate instability and fluctuations in price dynamics. In other terms, if agents are homogeneous with respect to their preferences and the processes they use for expectations formation, the presence of different trading behaviors based on different investment horizons has a minimum impact on the market stability. On the other hand, when agents are characterized by heterogeneous expectations about future market behavior, the heterogeneity in time horizons has a strong effect on the aggregate model behavior. In particular, the price dynamics turns out to be very sensitive to the way in which investors extrapolate their estimation of risk over time. To discuss this effect, we study two reasonable, albeit very simple, specifications for risk extrapolation and compare their effects on model dynamics.

One unexpected finding is that for a large region in the parameters space the increase of the investment horizon of fundamentalists leads to a *decrease* of the stability of the system. We argue in this paper that the ultimate reason of this phenomenon lies in the fact that the instability of the system is related with the relative demand of fundamentalists. If the relative demand is low (as it happens if they overestimate risk, for instance), then the system may loose its stability.

This Chapter is organized as follows. In the next Section we recall the importance of time horizons in investor's financial decision and briefly discuss previous, mostly classical, contributions. We also discuss there some alternative possibilities of the modeling of the investment horizons in the heterogeneous agent framework. In Section 2.3 the analytical model of market participation is introduced and various assumptions are discussed. We describe there two classes of agents, fundamentalists and chartists. The behavioral assumptions about these classes are made on the basis of empirical evidence. Moreover, we distinguish between two types of fundamentalists: sophisticated and unsophisticated. In Section 2.4 we discuss, as a first simple example, the model in the particular case in which there are no chartists in the market. We show that heterogeneity in time horizons itself is not enough to generate non-trivial price dynamics. In Section 2.5 we perform the study of the general case when both fundamentalists and chartist are on the market. A discussion of the implications of our findings with particular emphasis on the role of time horizons is performed in Section 2.6. Section 2.7 contains some final remarks and suggestions on possible future developments.

2.2 Time Horizons

Any financial advisor nowadays would start the formation of the portfolio for individual client with the question "What is the time horizon of your investment?" The individual time horizon, i.e. the plan for how long to maintain the market position, affects the investor's level of risk and, consequently, his portfolio choice. Moreover, different types of assets are not equally suitable for those who have long-term needs and for those who have short-term objectives.

Investors who have thirty years or more to invest in the market are usually thought of as those who have long time horizon. These investors are typically young professionals or even high school or college students. There is quite strong evidence that financial planners encourage such young investors to have a portfolio with greater risk comparing with other investors, which tends also to produce higher market returns over time. In contrast, those investors who are of pre-retiree and retiree status usually have short time horizon, less than five years. Investors who have short-term time horizons are generally the least tolerant of investment risk and are more concerned with preserving their existing capital and income. These two classes of investors judge risk very differently, and so it is not surprising that the optimal choice for these two groups can be different.

The necessity to distinguish between different types of traders' strategies is widely acknowledged. For example, pension or investment funds are usually thought as market participants with long investment horizons (at least, some years) who trade on the basis of fundamental information. On the contrary, speculators try to get profit over shorter periods and often use technical trading rules. For example, Frankel and Froot (1990) found that speculators on the foreign exchange market tend to use extrapolation of past trends to build the forecast on short horizons, while they revert to fundamental information to forecast long-run equilibrium. Thus, there are no doubts that on real markets traders with heterogeneous investment horizons interact. Consequently, it seems natural to investigate the influence of such interaction on the dynamics of the market.

Since the classical financial asset pricing models of Markowitz and Sharpe-Litner, known as the Capital Asset Pricing Model (CAPM), did not capture the problem of different time-horizons in investment decisions, the natural question about its generalization arose in the academic community. However, the intertemporal generalizations both of CAPM (LeRoy (1973), Merton (1973)) and of other asset pricing models (Samuelson (1969), Lucas (1978)) had extremely strict assumptions both about the preference structure of agents and the nature of returns dynamics. Indeed, the theoretical results about investment horizons[3] have been obtained entirely within the representative agent tradition.

In this paper we extend the analysis of the time-horizon issue bringing it within the agent-based framework, where the price dynamics is endogenously generated by the interaction of heterogenous boundedly rational agents. We will model the demand of agents through the mean-variance optimization. This procedure provides a demand function with reasonable properties and also allows to model the heterogeneity of the agents with respect to the two first moments of the return. There are different approaches to model a heterogeneous investment horizons in such setting.

One can assume, for instance, that the agent with a time horizon of $\eta > 0$ periods forward does not correct his portfolio between time t, when his decision about portfolio is made, and time $t + \eta$. This agent will participate in the market activity only once each η periods and will choose his portfolio composition maximizing the expected wealth utility at η period in the future. This assumption introduces some level of "irrationality" in the agent's behavior, since the agent is supposed to ignore the information revealed by the trading activity that takes place in the η trading sessions occurring between his consecutive market participations. On the other hand, this behavior may not be unrealistic, especially for individual investors.

Another, rather extreme, possibility is to assume that each agent participates in the market activity at each period, continuously correcting his portfolio composition in order to maximize utility as a function of future wealth, but taking into account the fact that the portfolio will be revised each period, on the basis of new information. Such an approach assumes a high level of rationality in agents description, and it has been widely used (in continuous-time case) after the seminal paper of Merton (1973). However, the Merton model, in general, is not analytically solvable even inside representative agent paradigm. The introduction of a certain degree of heterogeneity in agents behaviors would lead to even more complex dynamic programming problems that, in our opinion, would require a quite unrealistic degree of sophistication from the part of the agents.

In our model we follow a different, in a sense intermediate, strategy. We assume that an agent with time horizon η maximizes at period t his expected wealth at period $t + \eta$ without taking into account the possibility of future portfolio revisions. However, in each period the agent revises the portfolio if it is necessary, i.e. if the new market situation or his new expectations lead him to an optimal portfolio composition different from the one chosen in the last period[4]. Our formal model can, alternatively, by presented as an overlapping-generation model where each agents' type has η generations of traders with the same population. At each time period only one of these generations operates in the market. Let us now revert to the formal model implementation.

[3] See, for example, Samuelson (1969) who found the conditions under which long-horizon investors would make the same choice as short-horizon investors, and general discussion of other results in Campbell and Viceira (2002).

[4]This is similar with the strategies of the mutual funds which offer different investment plans for the specific time periods to individual investors and then operate in the financial markets from behalf of the clients. Fundamentalists in our model can be represented by such funds.

2.3 Model with Heterogeneous Expectations and Time Horizons

2.3.1 Market Dynamics

We consider an asset-pricing model with two assets: a risk-less asset (bond) that gives a constant interest rate $r_f > 0$ and a risky asset (equity) that pays a dividend at the beginning of each period t. In this model we will assume that the dividend is constant and denote it as D. The assumption of constant dividend allows us to concentrate on the issue of investment horizons in a situation as simple as possible. This assumption is satisfied, for example, in the bond market where the fixed coupon payment plays the role of the dividend. Moreover, Bottazzi (2002) demonstrated that model with constant dividend displays the same qualitative results as the same model in the case when the dividend is i.i.d. random variable with moderately small variance[5].

The bond is assumed to be the numéraire of the economy and its price is fixed to 1. The price of the risky asset P_t is determined each period on the basis of its total demand through a market clearing condition. The price return $\rho_{t,t+\eta}$ of the risky asset between time t and time $t+\eta$, without taking into account the dividend payments, reads

$$\rho_{t,t+\eta} = \frac{P_{t+\eta} - P_t}{P_t}.$$

Let W_t be the wealth of the agent at time t and let x_t be the share invested in the risky asset. Then the wealth of the agent in period $t+\eta$ as a function of return $\rho_{t,t+\eta}$ reads:

$$W_{t+\eta} = (1 - x_t)\, W_t\, (1 + r_f)^\eta + x_t\, W_t \left(1 + \rho_{t,t+\eta} + \frac{\hat{D}_\eta}{P_t}\right) \quad , \tag{2.3.1}$$

where $\hat{D}_\eta$ stands for the discounted stream of dividends paid to the agent from t to $t+\eta$. Thus, we assume that the agent fully reinvests the dividends in the risk-less bond. Let us introduce *fundamental price* of the risky asset as a discounted stream of the dividends. Since the dividend is constant the fundamental price reads

$$\bar{P} = \frac{D}{r_f} \quad , \tag{2.3.2}$$

so that one has $\hat{D}_\eta = \bar{P}\,((1+r_f)^\eta - 1) = \bar{P}\,R_\eta$. We also denote as $R_\eta = (1+r_f)^\eta - 1$ a positive return of the holding of the riskless asset during η periods.

At each period t the agent with time horizon η chooses the share of wealth to invest in the risky asset x_t in such a way to maximize the mean-variance utility of the future wealth defined

[5] Assumption of the constant dividend, i.e. absence of any fundamental risk, is not completely uncommon in the heterogeneous agent modeling. For instance, model in DeLong, Shleifer, Summers, and Waldmann (1990b) is also built under this assumption. In this respect, it is interesting to mention that the reason why the complex price dynamics can be generated in that model is different from the reason why it appears in our model. In the setting of DeLong, Shleifer, Summers, and Waldmann (1990b) all traders possess complete knowledge about market conditions and behave in completely rational way given this knowledge. The market can be, nevertheless, destabilized by the unpredictable noisy trade of part of participants. In our model, even if the market mechanism is known and all agents behave in deterministic way, the future market performance is deduced from the past by the agent, who are, therefore, only boundedly rational. The agents' heterogeneity creates a non-linear effect on the market, leading to possible unpredictable price movements.

in (2.3.1), thus, in principle, avoiding participation in the market until period $t+\eta$ *if the market behavior, in terms of price and returns dynamics, would not change.* However, new information about realized prices may force (and usually forces) agent to change the composition of his portfolio in next period. Thus, on the contrary to the dynamical programming approach, agents in our model do not take into account their future actions. In this sense, agents are boundedly rational.

The dynamics during one trading period proceeds as follows. Assume that in the beginning of time t agent n with investment horizon η_n possesses $A_{t-1,n}$ shares of the risky and $B_{t-1,n}$ shares of the risk-less assets. First of all, agent gets the dividends per each share of the risky asset and interest rate per each share of the risk-less assets. These financial streams are paid in terms of the numéraire. Then, for each notional price P, the agent, as a price-taker, makes some assumptions about the first two moments of the distribution of wealth $W_{t+\eta_n,n}$, conditional on: (i) his wealth at this period $W_{t,n}(P)$, and on (ii) the available information set $\mathfrak{I}_{t-1} = \{P_{t-1}, P_{t-2}, \dots\}$. Solving the following optimization problem:

$$\max_{x_t} \left\{ \mathrm{E}_{t,n}[W_{t+\eta_n,n}] - \frac{\beta_n}{2}\, \mathrm{V}_{t,n}[W_{t+\eta_n,n}] \right\} \tag{2.3.3}$$

subject to the budget constraint (2.3.1), agent computes the number of risky assets which he would like to have for a given notional price:

$$\tilde{A}_{t,n}(P) = \frac{x^*_{t,n}(W_{t,n}(P), P)\, W_{t,n}(P)}{P} \tag{2.3.4}$$

The excess demand function of the agent n is then $\Delta A_{t,n}(P) = -A_{t-1,n} + \tilde{A}_{t,n}(P)$. Given that all agents have formed their demand functions, the market clearing condition can be written as $\sum_{n=1}^{N} \Delta A_{t,n}(P) = 0$, where N is the total number of agents on the market. After when the price P_t is determined as the solution of the market clearing condition, agent n possesses $A_{t,n}$ shares of the risky asset, obtained according to (2.3.4), where for P we substitute P_t, and $B_{t,n} = P_t\,(A_{t-1,n} - A_{t,n}) + A_{t-1,n}\, D + B_{t-1,n}\,(1+r_f)$ shares of the risk-less assets. The economy is ready for the next round now.

2.3.2 Pricing Equation

In order to solve (2.3.3), agent n has to form expectations about his wealth at time $t+\eta_n$. Let us assume that agent n believes that the return $\rho_{t,t+\eta_n}$ has expected value $\mathrm{E}_{t,n}[\rho_{t,t+\eta_n}]$ and variance $\mathrm{V}_{t,n}[\rho_{t,t+\eta_n}]$. Then from (2.3.1), the conditional distribution of future wealth $W_{t+\eta_n,n}$, given current price P_t and wealth $W_{t,n}$, has the following expected value and variance:

$$\begin{aligned}
\mathrm{E}_{t,n}[W_{t+\eta_n,n}] &= W_{t,n}\,(1+r_f)^{\eta_n} + x_{t,n}\, W_{t,n} \left(\mathrm{E}_{t,n}[\rho_{t,t+\eta_n}] + \left((1+r_f)^{\eta_n} - 1\right) \left(\frac{\bar{P}}{P} - 1 \right) \right) , \\
\mathrm{V}_{t,n}[W_{t+\eta_n,n}] &= \left(x_{t,n}\, W_{t,n}\right)^2 \mathrm{V}_{t,n}[\rho_{t,t+\eta_n}] ,
\end{aligned}$$

where $\bar{P}$ stands for the fundamental price of the risky asset introduced in (2.3.2). Maximization of the mean-variance utility (2.3.3) with these expectations gives the share of wealth which agent n invests in the risky asset. According to (2.3.4) the demand for the risky asset of agent then reads:

$$\tilde{A}_{t,n}(P) = \frac{\mathrm{E}_{t,n}[\rho_{t,t+\eta_n}] + ((1+r_f)^{\eta_n} - 1)(\frac{\bar{P}}{P} - 1)}{\beta_n\, P\; \mathrm{V}_{t,n}[\rho_{t,t+\eta_n}]} \quad . \tag{2.3.5}$$

Thus, the demand for the risky asset is proportional to the excess return and inversely proportional to the expected risk. Demand is also decreasing function of the current price and of the agent's risk aversion coefficient β_n. Notice that the investment horizon appears in three places inside the demand function. First, it affects the expected excess return, since this return is have to be computed over η periods. Second, it affects the expected variance because of the same reason. And, third, it contributes to the wealth accumulation through the dividend re-investment.

The equilibrium[6] price P_t is determined implicitly as a solution of the market-clearing equation $\sum_{n=1}^{N} \Delta A_{t,n}(P) = 0$. For the agent-specific, possibly time dependent, quantities $x_{t,n}$ we introduce the following notation for the average of this variable over population of agents:

$$\left\langle x_{t,n} \right\rangle_n = \frac{1}{N} \sum_{n=1}^{N} x_{t,n} \quad .$$

In this notation, and under the assumption that the number of shares of the risky asset is constant and equal to A_{TOT}, we rewrite the market-clearing equation as the following quadratic equation

$$P^2 \cdot A - P \cdot B - C = 0 \quad , \tag{2.3.6}$$

whose coefficients read:

$$A = \frac{A_{TOT}}{N} \quad , \qquad B = \left\langle \frac{\mathrm{E}_{t,n}[\rho_{t,t+\eta_n}] - R_{\eta_n}}{\beta_n \, \mathrm{V}_{t,n}[\rho_{t,t+\eta_n}]} \right\rangle_n \quad \text{and} \quad C = \left\langle \frac{\bar{P}\, R_{\eta_n}}{\beta_n \, \mathrm{V}_{t,n}[\rho_{t,t+\eta_n}]} \right\rangle_n \ . \tag{2.3.7}$$

Thus, the pricing equation is quadratic in our model. Its first coefficient A represents the number of shares per capita. The second and third coefficients are averages of some agent-specific quantities. Remember that $R_{\eta_n} = (1 + r_f)^{\eta_n} - 1$ introduced in Section 2.3.1 stands for the return of the risk-less asset for agent n. Thus, the numerator of coefficient B is the expected *excess return* of the capital gain of the risky asset, while the numerator of the third term represents the capital which agent obtains from the stream of the dividends for the risky asset during η_n periods. Both coefficients are rescaled by the factor, which takes into account the agents' personal evaluation of the risk implied by keeping the risky stock in the portfolio.

The non-linear pricing equation (2.3.6) can be contrasted with linear equation for price determination in the model of Brock and Hommes (1998). Notice that we obtain a non-linear pricing equation due to the fact that we do not assume a zero total outside supply of the risky asset, i.e. we do not put restriction $\bar{A} = 0$. Since terms A and C are positive, (2.3.6) has two real roots, of which only one is positive, that we assume as the market price. Thus, our model has a unique and positive equilibrium price which reads

$$P_t = \frac{1}{2A} \left(B + \sqrt{B^2 + 4A\,C} \right) \quad . \tag{2.3.8}$$

As expected, price increases with expected capital gain and with the accumulated dividends. Increase in the investment horizon has unambiguously positive effect on the latter component.

[6] In order to avoid any misunderstanding, we stress here that the same word "equilibrium" is used in two circumstances in the context of the dynamical models in economics. Here we mean that at each time period the price is defined in such a way that demand is equal to supply, i.e. market is in Walrasian equilibrium. Afterwards, by "equilibrium" we will mainly mean the steady-state, i.e. fixed point, of the dynamical system describing the market evolution through time. We usually use word "dynamical" speaking about the latter type of equilibrium and word "temporary" speaking about the former type.

At the same time, price P_t decreases with the relative supply of the asset and with increase in the expected risk.

One important property of the price in (2.3.8) is its behavior in the homogeneous agent case when the constant price is expected. In such benchmark situation, for any agent n both $\mathrm{E}_{t,n}[\rho_{t,t+\eta_n}]$ and $\mathrm{V}_{t,n}[\rho_{t,t+\eta_n}]$ go to zero. However, the price can still be defined as the following limit

$$P_t = \lim_{\mathrm{V}_t[\rho_{t,t+\eta}]\to 0} \frac{-R_\eta + \sqrt{R_\eta^2 + 4\,A\,\bar{P}\,R_\eta\,\beta\;\mathrm{V}_t[\rho_{t,t+\eta}]}}{2A\beta\,\mathrm{V}_t[\rho_{t,t+\eta}]} \quad ,$$

where all the agent-specific indexes are omitted due to the homogeneity assumption. Simple derivation with the use of the l'Hospital's Rule shows that the last limit is equal to the fundamental price $\bar{P}$.

To close the model we only have to specify how beliefs are formed and evolve for different agents. In the next Section we will follow the standard approach of the agent-based literature of financial markets and consider various cases in which only few stylized types of traders participate in the market.

2.3.3 Types of Agents: Chartists and Fundamentalists

Since we focus on an analytical model, we will simplify the pricing equation (2.3.6). First of all, we assume that $\beta_n = \beta$ for all n. Second, we confine our attention to the case when only two types of agents trade in the market, and we call them, *chartists* and *fundamentalists*. Both types assume the existence of some underlying stochastic process describing the dynamics of price. Furthermore, we will distinguish two types of fundamentalists according to the rule which they use to extrapolate the prediction of next-period return variance on longer periods. In this section we specify how agents form their expectation about the mean and variance of future returns starting from their respective "visions of the world".

Chartists

Chartists assume that the future return can be predicted on the basis of the past history using some consistent statistical estimator. More precisely, at period t, in order to predict the average future return $\rho_{t,t+\eta}$, they use the exponentially weighted moving average (EWMA) estimator of return, y_{t-1}, based upon the available information (all past price realizations up to P_{t-1}). Analogously, for the prediction of the variance of the future return they compute the EWMA estimator of this variance, z_{t-1}, based upon the past variability of the return. Quantities y_{t-1} and z_{t-1} are defined as follows[7]:

$$y_{t-1} = (1-\lambda)\sum_{\tau=2}^{\infty}\lambda^{\tau-2}\rho_{t-\tau}\,,$$

$$z_{t-1} = (1-\lambda)\sum_{\tau=2}^{\infty}\lambda^{\tau-2}[\rho_{t-\tau} - y_{t-1}]^2 \quad .$$

Parameter $\lambda \in [0,1)$ measures the relative importance which agent puts on the past observations. The last available return has the highest weight, while others are declining geometrically

[7]To avoid the over-using of indexes we denote as $\rho_{t-\tau}$ the return for one period ahead, i.e. $\rho_{t-\tau,t-\tau+1}$.

in the past. Thus, the higher λ is, the less the relative weight of the recent observations. Therefore, one can interprete λ as a "memory" parameter. In extreme case, when $\lambda = 0$, agent uses the past return as the only possible value for the next return, and so the memory is the shortest possible. This is the case of "naïve" expectations. Notice that expressions above are analogous to the ones proposed by the RiskMetrics$^{\text{TM}}$ group (see RiskMetrics$^{\text{TM}}$ (1996)), and widely applied by real operators in their forecasting activity[8]. In what follows we will use the recursive form of the previous two equations:

$$\begin{aligned} y_{t-1} &= \lambda y_{t-2} + (1-\lambda)\rho_{t-2} \\ z_{t-1} &= \lambda z_{t-2} + \lambda\,(1-\lambda)\,(\rho_{t-2} - z_{t-2})^2 \end{aligned} \tag{2.3.9}$$

Equations (2.3.9) describe a one period ahead forecast. To build a long-term forecast, chartists assume that future price dynamics can be described as a geometric random walk. Then the forecasts for both expected returns and variances for η periods forward can be obtained by simply multiplying the one-period forecast by a factor η. Thus:

$$\mathrm{E}_{t,c}[\rho_{t,t+\eta}] = \eta\ \mathrm{E}_{t,c}[\rho_{t,t+1}] = \eta\, y_{t-1} \quad , \tag{2.3.10}$$

and

$$\mathrm{V}_{t,c}[\rho_{t,t+\eta}] = \eta\ \mathrm{V}_{t,c}[\rho_{t,t+1}] = \eta\, z_{t-1} \quad . \tag{2.3.11}$$

Fundamentalists

Fundamentalists believe that the price is governed by dynamics which constantly revert towards the stock fundamental value. The future price P_{t+1} will be, on average, between the current price and fundamental price $\bar{P}$, so that:

$$\mathrm{E}_{t,f}[P_{t+1}] = P_t + \theta\,(\bar{P} - P_t) \quad , \tag{2.3.12}$$

where $\theta \in [0, 1]$ describes the belief of fundamentalists about how reactive the market is in recovering the fundamental price. Thus, fundamentalists believe that if the asset is undervaluated, then the price of it will increase, whereas when the asset is over-valuated, then the price will fall[9]. In the case when $\theta = 0$ equation (2.3.12) gives so-called "naïve" expectations $\mathrm{E}_{t,f}[P_{t+1}] = P_t$, while $\theta = 1$ corresponds to the case when fundamentalists believe that the fundamental value will be already reached in the next period. The expression of the multi-steps expected return under the assumption of mean-reverting market dynamics described by (2.3.12) reads

$$\mathrm{E}_{t,f}[\rho_{t,t+\eta}] = \Big(1 - (1-\theta)^{\eta}\Big)\left(\frac{\bar{P}}{P_t} - 1\right) \tag{2.3.13}$$

This result can be straightforwardly obtained by the recursive use of equation (2.3.12).

Since the volatility of the asset is determined essentially by the opinion of the market, we assume that fundamentalists build their forecast for the return volatility one period ahead in the same way as chartists do, i.e. $\mathrm{V}_{t,f}[\rho_{t,t+1}] = z_{t-1}$.

[8]The RiskMetrics$^{\text{TM}}$ group actually proposes an EWMA estimator of the volatility, defined as the second moment of the returns distribution. The expression above represents its natural extension to the central moment.

[9]Strictly speaking, price P_t does not belong to the information set. Thus, equation (2.3.12) describes prediction of fundamentalists about the future price *behavior* with respect to the price level today

At this stage, when we have to specify the fundamentalists' forecast $\mathrm{V}_{t,f}[\rho_{t,t+\eta}]$ for the variance of return after η periods, we will distinguish between two types of fundamentalists. We will refer to one type as *sophisticated* fundamentalists and to the other type as *unsophisticated* fundamentalists, according to the complexity of the analytical tool which they use to compute their forecast.

Unsophisticated Fundamentalists. These agents use relatively simple reasoning, assuming like chartists that the variance of return linearly increases with time and so:

$$\mathrm{V}_{t,f}[\rho_{t,t+\eta}] = \eta\ \mathrm{V}_{t,f}[\rho_{t,t+1}] = \eta\, z_{t-1} \quad . \tag{2.3.14}$$

On the one hand, as we show below, this assumption about fundamentalists' forecast contradicts to the expectations about price behavior in (2.3.12). On the other hand, this specification of the variance corresponds to behavior commonly found among financial investors who tend to use fundamental evaluation in judging different investment opportunities while preferring an econometric (technical) approach for the evaluation of the implied risk. Moreover, the Brownian scaling of the volatility described in (2.3.14) is qualitatively similar to the one actually found for empirical markets (Dacorogna, Gençay, Müller, Olsen, and Pictet, 2001).

Sophisticated Fundamentalists. These fundamentalists use a more sophisticated way of forecasting the variance, which is consistent with their return expectations. They believe that (2.3.12) describes the price dynamics *in each period*. This assumption allows them to model the forecast of time series of future returns as a mean reverting stochastic process. In Appendix B.1 we derive the Fokker-Planck equation that describes the evolution of the long-horizon forecast and we show that, first, the prediction for the long-term return satisfies (2.3.13), and, second, that the long-term asset volatility forecast can be obtained from the one-period volatility forecast $V_{t,f}[\rho_{t,t+1}] = z_{t-1}$ according to

$$\tilde{\mathrm{V}}_{t,f}[\rho_{t,t+\eta}] = \frac{1-(1-\theta)^{2\eta}}{\theta(2-\theta)}\, z_{t-1} \tag{2.3.15}$$

The crucial difference between (2.3.14) and (2.3.15) consists in the fact that the former infinitely increases with fundamentalists' time horizon η, while the latter converges to the asymptotic value $z_{t-1}/(\theta(2-\theta))$. Thus, on the large investment horizons, *unsophisticated fundamentalists overestimate risk*. We will see later that this difference leads to crucial change in the dynamics of the model.

The specification of agents' behaviors introduced in this Section, concludes the building of the model of artificial market. In the next Section we address one of the central questions of this Chapter: Is the assumption of heterogeneity in investment horizons enough to generate dynamics of price different from the convergence to the fundamental value? We will start by considering the situation of the market without chartists. In Section 2.5 we continue this analysis and extend it to the situation in which both chartists and fundamentalists participate in the market.

2.4 Market without Chartists

This Section is devoted to the case when the market is populated only by fundamentalists. We explicitly provide the analysis for the case when all fundamentalists are unsophisticated. One can easily check that our results do not change in the case when fundamentalists are sophisticated.

2.4.1 Homogeneous Time Horizons

Let us start with the simplest situation and assume that on the market there are only non-sophisticated fundamentalists with the same time horizon η. They use (2.3.13) and (2.3.14) as forecasting rules. Then pricing equation (2.3.6) becomes:

$$\eta\beta\bar{A}\,P^2\,z_{t-1} + P\Big((1+r_f)^\eta - (1-\theta)^\eta\Big) - \bar{P}\Big((1+r_f)^\eta - (1-\theta)^\eta\Big) = 0 \quad . \tag{2.4.1}$$

Analogously to (2.3.8), price P_t of the market in period t is provided as the positive solution of this quadratic equation. It reads:

$$P_t = \frac{-r + \sqrt{r^2 + 4\,d\,\gamma\,z_{t-1}}}{2\,\gamma\,z_{t-1}} \quad ,$$

where the following set of parameters has been introduced

$$\begin{aligned}
\gamma &= \eta\beta\bar{A} \quad , \\
r &= (1+r_f)^\eta - (1-\theta)^\eta \;=\; (r_f+\theta)\,B_\eta(R,\theta) \quad , \\
d &= r\bar{P} \;=\; D\,(1+\theta/r_f)B_\eta(r_f,\theta) \quad ,
\end{aligned}$$

and polynomial B_η reads: $B_\eta(r_f,\theta) = \big((1+r_f)^\eta - (1-\theta)^\eta\big)/(r_f+\theta)$.

Using the pricing equation together with (2.3.9), scaling the price as follows $p_t = \gamma\,P_t$ and introducing $s = d\gamma$, we get the following 3-dimensional system describing the dynamics of the market:

$$\begin{cases}
p_{t+1} = f(z_t) = \dfrac{-r + \sqrt{r^2 + 4sz_t}}{2z_t} \\[2ex]
y_{t+1} = \lambda y_t + (1-\lambda)\left(\dfrac{f(z_t)}{p_t} - 1\right) \\[2ex]
z_{t+1} = \lambda z_t + \lambda(1-\lambda)\left(\dfrac{f(z_t)}{p_t} - 1 - y_t\right)^2
\end{cases} \tag{2.4.2}$$

The properties of this system are discussed in Appendix B.2. We show there that the system has only one fixed point, which is locally[10] stable for all $\lambda < 1$. Thus, the price converges to the fundamental value independently of the value η. In fact, changes in η lead to changes in the values of parameters s and r (both of them are increasing together with η), but not to the modifications of the system itself. The speed of convergence to the fixed point depends, however, on η.

[10] The simulations with different parameters and initial values suggest that the point is, probably, even globally stable.

2.4.2 Heterogeneous Time Horizons

Consider now the market composed of N fundamentalists: N_1 of them have the investment horizon equal to 1, and N_2 have the horizon equal to 2.[11] If we denote the shares of different types of fundamentalists as f_1 and f_2 (so $f_i = N_i/N$ for $i = 1, 2$), the market clearing condition reads:

$$2\,\beta\,\bar{A}\,P^2\,z_{t-1} + (P - \bar{P})\,\Big(2\,f_1\,(r_f + \theta) + f_2(r_f + \theta)\,B_2(r_f + \theta)\Big) = 0 \quad .$$

There is an obvious similarity between (2.4.1) and the last equation. Indeed, we have the same dynamical system (2.4.2) with the following values of parameters:

$$\begin{aligned}
\gamma &= 2\beta\bar{A} \quad , \\
r &= (r_f + \theta)(2f_1 + f_2 B_2) = (r_f + \theta)(2 - f_2\theta) \quad , \\
d &= r\bar{P} \quad .
\end{aligned}$$

and the asymptotic behavior of the system remains the same. Thus, in the case of two groups of fundamentalists with different investment horizons, the system converges to the fundamental steady-state. In other words, in the market with fundamentalists the heterogeneity in the investment horizons does not affect the stability of the system.

In this Section we have shown that the dynamical system does not change with the introduction of different groups of fundamentalists having different time horizons and the dynamics remains the same independently from the initial conditions. Thus, we can conclude that there is no difference in the dynamics between the case with homogeneous (with respect to time horizon) fundamentalists and the case with heterogeneous ones. In both cases the price converges to the fundamental value and expectation about variance goes to 0.

This result is similar to a classical result of the financial literature, even if we obtained it using a completely different approach. Indeed, the solution of the dynamic programming model of Merton (1973) in the case of homogeneous expectations and under specific assumption about preferences (power utility function) shows that the optimal choice of the investor, in each period, does not depend on his horizon. This implies that the dynamics of the model remains the same even if agents have different time horizons. We reach the same qualitative conclusions, using a different utility function and in the presence of heterogenous agents.

2.5 Complete Model: Fundamentalists vs. Chartists

The qualitative dynamics of the price in the previous model did not depend on the parameters: price always converged to the fundamental value. This result crucially depends on the assumption about how agents form expectations. In this Section we consider the complete model where both fundamentalists and chartists are present. We present here only the specification of the deterministic system describing market dynamics in such situation. The analytical investigation of the stability of the system is presented in the Appendix, while the discussion of the results is postponed to the next Section. We start with the case when all fundamentalists are non-sophisticated and in the last subsection consider the case with sophisticated fundamentalists. First of all, we consider the simplest benchmark case where all

[11] The analysis can be straight-forwardly generalized to the case with two and more groups of fundamentalists possessing arbitrary time horizons.

agents have the same (one period) time horizon and later go to the generalization on the case of different time horizons.

2.5.1 Benchmark: Common Investment Horizon

Consider the market composed of N_1 unsophisticated fundamentalists who make predictions according to (2.3.13) and (2.3.14), and N_2 chartists who make predictions according to (2.3.10) and (2.3.11). Denote as f_1 and f_2 the fractions of fundamentalists and chartists, correspondingly, and assume that all investors on the market have the same time horizon equal to 1. The same situation is studied at length in Bottazzi (2002).

Since the rule for the formation of the expected variance is the same for all agents, the market clearing condition reads:

$$\beta \bar{A} P^2 z_{t-1} + P\left(r_f + f_1\theta - f_2 y_{t-1}\right) - \bar{P}\left(r_f + f_1\theta\right) = 0 \quad .$$

The positive root of this equation, together with (2.3.9) gives the following 3 dimensional system:

$$\begin{cases} p_{t+1} = g(y_t, z_t) = \left(y_t - r + \sqrt{(y_t - r)^2 + 4sz_t}\right)/2z_t \\ y_{t+1} = \lambda y_t + (1-\lambda)\left(\dfrac{g(y_t, z_t)}{p_t} - 1\right) \\ z_{t+1} = \lambda z_t + \lambda(1-\lambda)\left(\dfrac{g(y_t, z_t)}{p_t} - 1 - y_t\right)^2 \end{cases} \tag{2.5.1}$$

where as in (2.4.2) we have $p(t) = \gamma P_t$ and $s = d\gamma$, and the parameters are introduced as follows

$$\begin{aligned} \gamma &= \beta\bar{A}/f_2 \quad , \\ r &= (r_f + f_1\theta)/f_2 \quad , \\ d &= r\bar{P} \; = \; (D + f_1\theta\bar{P})/f_2 \quad . \end{aligned} \tag{2.5.2}$$

In Appendix B.3 we show that the only fixed point of the system is $(\gamma\bar{P}, 0, 0)$. The sufficient condition for the local asymptotical stability of the system reads:

$$r + \lambda > 1 \quad , \tag{2.5.3}$$

where parameter r was defined in (2.5.2). With decreasing λ and/or r the system loses stability when complex eigenvalues cross the unit circle. In other words, the system exhibits Hopf bifurcation.

2.5.2 Generalization: Different time horizons

Now we assume that all unsophisticated fundamentalists have the same time horizon η_1 and all chartists have the same time horizon η_2. This modification of setup does not change system (2.5.1) but leads only to changes of the values of the parameters. To see this we again write the market clearing condition:

$$\begin{aligned} \eta_1\eta_2\beta\bar{A}\, P^2 z_{t-1} \;&+\; P\left(\eta_2 f_1(r_f+\theta)B_{\eta_2}(r_f,\theta) - \eta_1\eta_2 f_2\, y_{t-1} + \eta_1 f_2 r_f B_{\eta_2}(r_f,0)\right) \\ &-\; \bar{P}\left(\eta_2 f_1(r_f+\theta)B_{\eta_2}(r_f,\theta) + \eta_1 f_2 r_f B_{\eta_2}(r_f,0)\right) = 0 \quad , \end{aligned}$$

where polynomial B_η was defined in Section 2.4.1. It reads

$$B_\eta(r_f,\theta) = \frac{(1+r_f)^\eta - (1-\theta)^\eta}{r_f+\theta} \quad .$$

Thus, we obtain the same system as (2.5.1) but with parameters

$$\begin{aligned} \gamma &= \frac{\beta\bar{A}}{f_2} \quad , \\ r &= \frac{1}{\eta_1}\frac{f_1}{f_2}(\theta+r_f)B_{\eta_1}(r_f,\theta) + \frac{1}{\eta_2}r_f B_{\eta_2}(r_f,0) \quad , \\ d &= r\bar{P} \quad . \end{aligned} \tag{2.5.4}$$

Since the system remains the same, it still has the only fixed point $(\gamma\bar{p},0,0)$. The condition for the local stability of that point is given by (2.5.3), however, parameter r is now defined in (2.5.4).

2.5.3 Case of sophisticated fundamentalists

Now we can turn to the case when fundamentalists on the market are sophisticated, i.e. when they use predictions for the variance provided by expression (2.3.15) instead of (2.3.14). In both cases considered above the overall effect of this modification will amount to the redefinition of parameters values. It leads to the change in dynamics for given parameters value. Indeed, it is easy to check that with the new forecasting rule the parameters of the system (2.5.1) become

$$\begin{aligned} \gamma &= \frac{\beta\bar{A}}{f_2} \quad , \\ r &= \frac{\theta(2-\theta)}{1-(1-\theta)^{2\eta_1}}\frac{f_1}{f_2}(\theta+r_f)B_{\eta_1}(r_f,\theta) + \frac{1}{\eta_2}r_f B_{\eta_2}(r_f,0) \quad , \\ d &= r\bar{P} \quad . \end{aligned} \tag{2.5.5}$$

2.6 Discussion

The results of the previous Section suggest that the investment horizons do affect the market dynamics. We discuss here this issue in details.

Unsophisticated Fundamentalists

Let us focus, first, on the case when fundamentalists are unsophisticated. The relevant set of parameters in this case given by (2.5.4). The analysis of condition (2.5.3) and the dependence of r in (2.5.4) with respect to the different parameters shows that the stability region in the parameter space of the fixed point becomes larger with

- the "smoothness" of the agents' forecasting behavior λ (in other words, with the length of the agents' memory);
- the share of fundamentalists on the market f_1;

- the risk-less return r_f;
- the perception of fundamentalists about the efficiency of the market θ.

These results are quite intuitive and do not deserve further description. Their emergence can be traced back to the assumptions about agents behavior and market structure. Let us now turn to the analysis of the role of investment horizons.

First of all, notice that an increase of the time horizon of each group of traders presented in the market alone do not disturb the stability of the system. This we have shown in Section 2.4 for the case of fundamentalists. The same happens also in the market with only chartists. Moreover, from (2.5.4) it follows that parameter r increases with η_2. Therefore, *the region of the stability of the system enlarges with increase of the investment horizon of chartists.* It happens both with and without fundamentalists in the market.

At the same time, r in (2.5.4) depends on η_1 in a non-monotonic way. Namely, r as a function of η_1 displays an U-shaped behavior (see Fig. 2.1) reaching its minimum in the point, which represents the solution of the following equation:

$$(1+r_f)^{\eta_1}(\eta_1 \ln(1+r_f)+1) = (1-\theta)^{\eta_1}(\eta_1 \ln(1-\theta)-1) \quad .$$

This fact is responsible for the non-monotonic way in which the increase of the value of time horizons of agents influences the dynamics of the system. We plot in Fig. 2.2 the price dynamics obtained with a set of parameters (see caption) that in the benchmark case (when all agents have time horizons $\eta_1 = \eta_2 = 1$) generates convergence toward the fundamental value. The only difference is the increasing of time horizon for fundamentalists, we choose $\eta_1 = 18$. This graph illustrates also one of the typical non-converging pattern of price generated by the system (2.5.1) – price follows periodic behavior: after relatively slow rise in price a sudden fall happens. This behavior reminds the crashes after "speculative bubbles" found in real financial markets.

Fig. 2.3 is a bifurcation diagram for the price, when one changes the time horizon of fundamentalists η_1 (all other parameters have the same values as in Fig. 2.2). As we can see, with an increase of η_1 the system looses stability and goes to the region with dynamics similar to Fig. 2.2. However, with further increasing of η_1 the fundamental value becomes stable again.

To better understand the dependence of the stability of fixed point on the time horizons we plot in Fig. 2.4 the graph of the parameter r in (2.5.4) as a function of both η_1 and η_2. For the fixed value of λ, the region of stability is determined by an intersection of the surface on the picture with the horizontal plane at height $1-\lambda$. We project the curve in this intersection and show it on the same graph. The curve with parabolic shape in Fig. 2.4 gives the boundary of the stability region of the fixed-point in the space of parameters (η_1, η_2).

To summarize, the time horizons of fundamentalists play, in a certain sense, bigger role than the time horizons of chartists. Even if the increase of both parameters will eventually drive the system in the region of stability, for small enough values *increase of the time horizon of fundamentalists can break the stability of the fixed point, while an increase of the time horizon of chartists can not.* Before to explain this result, let us consider the case when all fundamentalists are sophisticated.

Sophisticated Fundamentalists

In order to analyze the stability of the system in this case we have to look at the definition of the parameter r given in (2.5.5). Notice that the first term for r does no longer have the

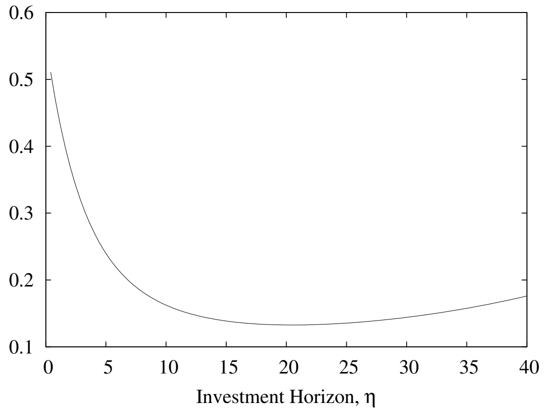

Figure 2.1: To illustrate a non-monotonic dependence of parameter r on the fundamentalists' investment horizon we demonstrate the typical shape of function $((1+r_f)^\eta - (1-\theta)^\eta)/\eta$. (Computed with $r_f = 0.05$ and $\theta = 0.4$.) With η large, the function increases exponentially to the infinity.

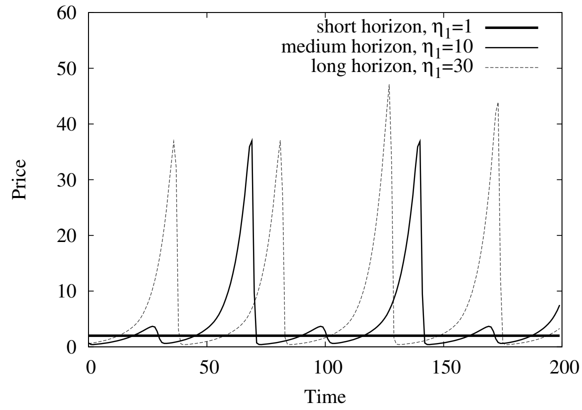

Figure 2.2: The long-run price dynamics generated by system (2.5.1) for different investment horizons of fundamentalists. With $\eta_1 = 1$ the price converges to the fundamental value, but with $\eta_1 = 10$ or $\eta_1 = 30$ the price fluctuates around that value with periodic crushes and booms. Parameters are $r_f = 0.05$, $D = 0.1$, $\bar{A} = 2$, $\beta = 1$, $\lambda = 0.85$, $f_1 = 0.2$, $\theta = 0.4$, $\eta_2 = 1$. Initial conditions are $P = 1$, $y = 0.01$, $z = 0.0001$.

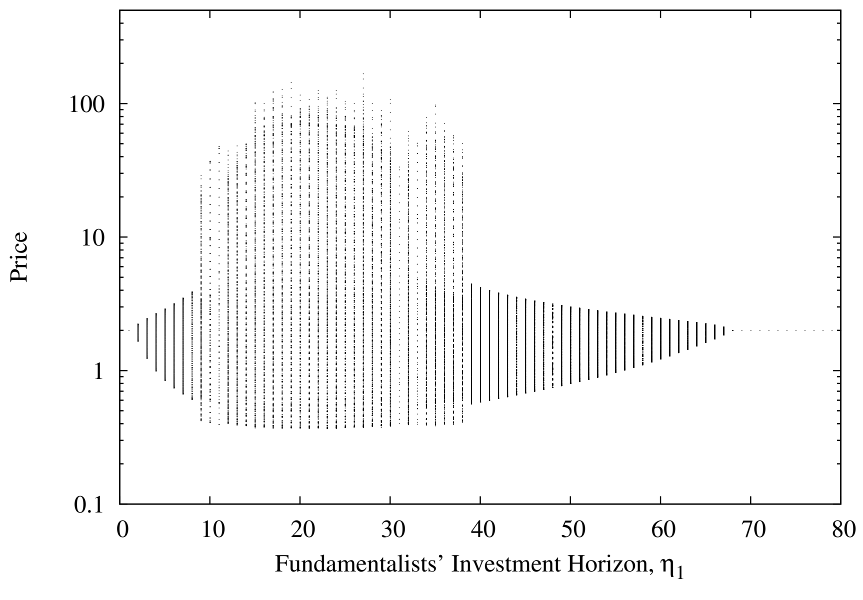

Figure 2.3: Bifurcation diagram. The price support of a 1000 steps orbit (after a 1000 steps transient) is shown in the log-scale for different values of η_1 from 1 to 80. (The values of parameters for simulation: $r_f = 0.05$, $D = 0.1$, $\bar{A} = 2$, $\beta = 1$, $\lambda = 0.85$, $f_1 = 0.2$, $\theta = 0.4$, $\eta_2 = 1$). The initial conditions are $P = 1$, $y = 0.01$, $z = 0.0001$.

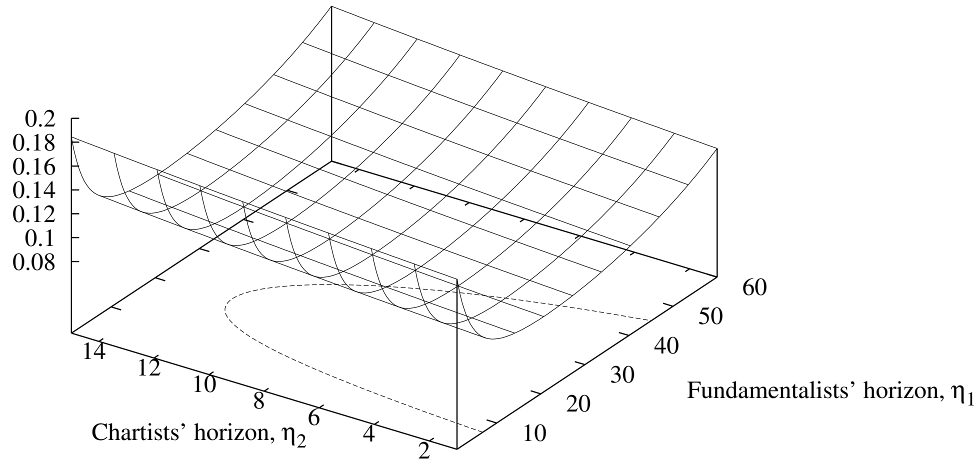

Figure 2.4: Parameter r as a function of η_1 and η_2. The values of the other parameters in (2.5.4) are $r_f = 0.05$, $f_1 = 0.2$, $\theta = 0.3$. The curve on the horizontal (η_1, η_2)-plane confines the closed region, where the fundamental price is an unstable fixed point for $\lambda = 0.9$.

U-shape behavior as a function of η_1. This implies that, in contrast to the previous situation, the increase of the time horizons of fundamentalists cannot lead to unstable dynamics. In other words, the situation shown in the bifurcation diagram in Fig. 2.3, where the increase of the time horizon η_1 brings the system outside the stable region, cannot occur.

Comparing prediction rule for the long-term variance given by (2.3.14) with prediction according to (2.3.15), we can conclude that the source of the unstable effect of the increase of the fundamentalists' time horizon is the over-estimation by unsophisticated fundamentalists of the risk of asset in medium run (near 10 periods). Let us discuss this, seemingly paradoxical result, in more detail.

Two effects of change in the investment horizon

Our system generates either dynamics converging to the fixed point or unstable trajectories with consequent bubbles and crashes, like we illustrated in Fig. 2.2. In order to explain our findings concerning the investment horizons, we consider the benchmark situation in which both fundamentalists and chartists are myopic, and price is growing, so that the economy is in the bubble path. As Fig. 2.5 illustrates, in this situation the growth rate of price reduces due to the fact that the expected variance z_t approaches zero with decreasing rate. Therefore, the expected (by chartists) return y_t decreases as well and so (from (2.3.5)) the demand decreases. Eventually, price starts to decrease and the bubble bursts.

Notice that in such situation the behavior of two types of agents is completely opposite. As the middle panel of Fig. 2.5 demonstrates, chartists expect the positive return on the bubble path, and, therefore contribute to the continuation of the price trend. On the other hand, fundamentalists' demand affects the price in the opposite way, since price in the bubble path is greater than fundamental price $\bar{P}$. Thus, fundamentalists bring the stability to the system, while chartists create unstable momentum.

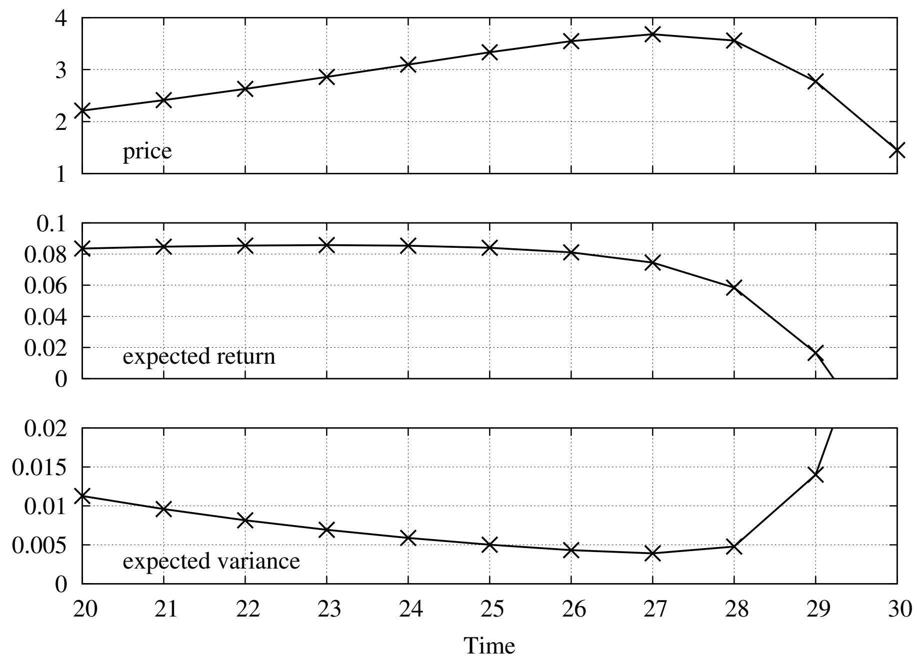

Figure 2.5: Time series of price (**Upper Panel**), expected return (**Middle Panel**) and expected variance (**Lower Panel**) computed according to (2.3.9) during 11 consequent periods. The value of parameters are the same as in Fig. 2.2 with time horizon of fundamentalists equal to 10.

Let us consider the effects of the investment horizons on the demand equation (2.3.5). It reads for an arbitrary trader:

$$\tilde{A}_t(P) = \frac{\mathrm{E}_t[\rho_{t,t+\eta}] + \big((1+r_f)^\eta - 1\big)\big(\frac{\bar{P}}{P} - 1\big)}{\beta\, P\ \mathrm{V}_t[\rho_{t,t+\eta}]} \quad .$$

Demand depends on the expected return for the risky asset which all traders scale linearly with the time horizon. It also depends on the excess return of the dividend yield with respect to the riskless asset. This excess return is negative when the economy is on the bubble path. With increase of the time horizon, this term grows exponentially. Finally, demand depends on the expected variance. The variance is scaled either linearly, when the agent is chartist or unsophisticated fundamentalist, or, in accordance with (2.3.15), less-than-linearly if the agent is sophisticated fundamentalist.

Now it is straight-forward to see that due to such scalings, the demand of *any* trader decreases when his investment horizon increases. Decrease of the demand is, clearly, stabilizing factor for the system on the bubble path, since it leads to the slowing down of the growth rate of price. Then, how can we explain the phenomenon illustrated in Fig. 2.3? Why, in the case of unsophisticated fundamentalists, the stability region in parameter space can even be shrinked?

Our explanation is the following. The non-linear pricing equation (2.3.6) depends on the weighted demand of the different groups of traders. The stability/instability of the dynamics depends, therefore, not only on the total demand but also on the relative impact of fundamentalists (which contribute to the stabilization of the system) and chartists (who creates the opposite effect). Therefore, the stabilizing *total demand effect*, which we found above is not the only one, which affects the dynamics when the investment horizons increase. The second effect is the *substitution effect* of the demand structure.

When the time horizon of fundamentalists increases, the relative share of them in total demand *decreases*. Thus, the substitution effect plays destabilizing role in this case, and the overall effect depends on the relative size of two effects. If fundamentalists form the long-term forecast for the variance according to (2.3.15), which increases with η_1 slower than (2.3.14), then the substitution effect turns out to be not strong enough to destabilize price dynamics.

Summarizing, contrary to the cases when market is populated by the agents from one single group (fundamentalists or chartists), the time horizon of investors does influence the dynamics of price in the situation with mixture of two groups. Moreover, the expression for r in (2.5.4) shows that the change in fundamentalists' time horizons leads to qualitatively different consequences than the changes in chartists' time horizons. In other words, the effect of changes in time horizons depends on the whole ecology of agents. Finally, comparative analysis of the situations when one of the groups increases their investment horizon shows that there are two effects relevant for the stability. In the case when fundamentalists increase investment horizons, these effects are opposite. In this case, if the substitution effect of relative demand is strong enough, unstable dynamics can be generated.

2.7 Conclusions

In this part of the thesis we analyzed an agent-based model with two sources of heterogeneity. First, agents possess different horizons for their investments, and, second, they have different expectations about future returns.

We found that in the world of fundamentalists, the sole introduction of heterogeneity in time horizons is not enough to generate non-trivial price dynamics. However, when the expectations of agents are heterogeneous, the resulting dynamics can be strongly affected by their time horizons.

As already noted in Bottazzi (2002), the apparently "harmless" hypothesis of describing traders as utility-maximizing agents updating their expectations on the past market history can lead to huge movements in price and to high degree of "inefficiency". This shows that the notion of equilibrium expressed by the Efficient Market Hypothesis is, in fact, extremely weak and can be made unstable with very mild assumption about agents' behavior.

The main conclusion of this Chapter is the emergence of non-trivial effects of different time horizons on the market price. First of all, we find that minor changes in the time horizons of some subpopulation of the agents may lead to large qualitative changes in the dynamics of price. Second, we see that whether and how the dynamics changes strongly depends on the kind of this subpopulation. Third, the effect of changes in time horizons depends on the ecology of agents. Finally, the dynamics depends on the way in which the agents estimate risk.

This model represents only a first step in the study of the influence of heterogeneity in time horizons on the price dynamics. One of the possible further directions could be the simulation of the model with a large number of different classes of agents within the general framework that we presented in Section 2.3.2.

Part II

Model with Endogenous Wealth Dynamics

Chapter 3

Market Evolution under Endogenous Wealth Dynamics

In this Chapter we proceed to the second theme of the thesis. We build here an agent-based model in which an interplay between aggregate price dynamics and micro behavior of heterogeneous agents occurs through the wealth evolution. The intuitive idea underlying our approach can be described as an interaction of two feedback mechanisms.

3.1 Two Feedback Mechanisms

A first feedback mechanism, prevalent inside the heterogeneous agent modeling, we have already encountered in the model discussed in Chapter 2. It emerges when agents in their investment decisions deliberately take into account market dynamics in the past. In dynamic model with endogenous price determination it implies that past return realizations affect future return. Thus, the first feedback mechanism applies to the *return* dynamics and appears due to its endogenous implementation.

A second feedback mechanism emerges because of the following reasoning. If we accept that the individual decisions affect the market price, then there is no reason to believe that the impact of these decisions is identical for different agents. In particular, those agents who possess greater wealth will, arguably, invest greater amount of money, so that their demand will have a greater impact on the market price. In dynamical framework it implies that those agents who have been more successful in the previous stages of the market competition enjoy greater power in the later price determination. The success of an agent here is defined in terms of his wealth return, so that one can immediately recognize a feedback loop in the dynamics of *wealth distribution*: the shape of the wealth distribution affects the price dynamics, which in turn influence the relative success of different investors and therefore changes the wealth distribution. This mechanism has been studied at length inside, so-called, evolutionary finance literature.

The idea of our model is to consider these interrelated feedback effects together and study the market dynamics in the presence of both of them. There have been only very few analytical attempts in this direction. Indeed, as the discussion in the next Section will illustrate, most of the heterogeneous agent models ignore the second feedback due to the specific assumption on the agents' demand. On the other hand, the first feedback is not taken into account inside the evolutionary financial literature, where the return is assumed to be exogenous. To our knowledge only the models in Cabrales and Hoshi (1996), Chiarella and He (2001, 2002a) and

Chiarella, Dieci, and Gardini (2004) incorporate both these feedback mechanisms.

It is remarkable that one of the first *simulation* models of financial market, developed by Haim Levy, Moshe Levy and Sorin Solomon, is built in the framework where both wealth and price are determined endogenously, so that both feedbacks are working. The achievements of this simulation approach have been summarized in the book Levy, Levy, and Solomon (2000) about which one of the founders of the modern financial theory, the Nobel Laureate Harry Markowitz said

> Levy, Solomon and Levy's Microscopic Simulation of Financial Markets points us towards the future of financial economics. If we restrict ourselves to models which can be solved analytically, we will be modeling for our mutual entertainment, not to maximize explanatory or predictive power.

This quote suggests that it was an analytical complexity which prevented the rigorous investigations of the model with both price and wealth feedback mechanisms. The remaining part of this thesis will try to dispel the suspicions about the impossibility to analyze such framework analytically.

The development of the analytical counterpart of any simulation model is not a goal *per se*, however. It should be, instead, considered as an important exercise in the direction of better understanding of the scope of the model and robustness of its findings obtained through the simulations. For the Levy, Levy and Solomon model the necessity of such exercise is further confirmed by some discrepancies between original and followed results as summarized in the conclusion of paper Zschischang and Lux (2001):

> This paper has reported some findings from our re-investigation of the Levy, Levy and Solomon model of a microscopic stock market. While some of these may provide interesting additional details (e.g., the perplexing sensitivity of the long-term outcome to slight changes in initial conditions), other findings may serve to question some earlier results (e.g., with respect to the chaotic properties of certain time paths). However, from our (the economists') point of view, the most interesting new result is the failure of the strategies of the original Levy, Levy and Solomon model in the presence of traders with the simple device of holding shares as a constant fraction of their wealth.

This Chapter is organized as follows. In the next Section we explain why the feedback mechanism functioning through the agents' wealth dynamics is not emerging in the most of the agent-based analytical contributions. In technical terms, we discuss the issue of the demand specification, demonstrating the differences between Constant Absolute Risk Aversion (CARA) and Constant Relative Risk Aversion (CRRA) frameworks. We also present an experimental evidence in favor of the latter. In Section 3.3 we review some of the previous contributions inside CRRA framework including evolutionary finance models and microscopic simulations of Levy, Levy and Solomon. The problems which one encounters trying to model the CRRA behavior analytically are discussed in Section 3.4 with some examples from four models of Cabrales and Hoshi (1996), Chiarella and He (2001, 2002a) and Chiarella, Dieci, and Gardini (2004). This discussion will lead us to the description of the ideas underlying our approach.

We start the formal implementation of these ideas in Section 3.5 where the basic equations of our model are derived. We show that these equations define an algebraic system in terms of price and wealth. The solution of this system together with necessary and sufficient condition for its existence is presented in Section 3.6. In Sections 3.7 and 3.8 we close the model by providing our specification of, respectively, exogenous dividend process and generic, endogenous agents' behavior. We conclude this Chapter by Section 3.9 outlining those steps which will be undertaken in the subsequent Chapters.

3.2 On the Demand Specification

The implementation of the feedback mechanism linking the wealth dynamics to the price evolution depends on the specification of the agent's demand. It is, indeed, the peculiar property of the demand function that prevents the presence of such mechanism in the numerous analytical models including Brock and Hommes (1998) and also in the simulation model in Arthur, Holland, LeBaron, Palmer, and Tayler (1997). Let us, first, illustrate this point and, then, discuss alternative demand definitions.

3.2.1 Mean-Variance Utility Maximization

We consider the following investment problem in a two-asset economy for myopic agent who uses mean-variance utility optimization:

$$\max_{x_t} \left\{ \mathrm{E}[W_{t+1}] - \frac{\beta}{2}\, \mathrm{V}[W_{t+1}] \right\} \qquad \text{s.t.} \quad W_{t+1} = W_t\,(1+r_f) + x_t\, W_t\,(\rho_{t+1} - r_f) \quad . \tag{3.2.1}$$

This is the same problem which we have been encountered in Chapter 2. As usually, W_t and x_t denote, respectively, the agent's wealth and investment share at time t; while r_f and ρ_{t+1} denote the returns of the riskless and risky assets, correspondingly. Positive coefficient β captures the agent's attitude towards risk. The incomplete list of both analytical and simulation models which are micro-founded by such maximization problem contains the contributions in Grossman and Stiglitz (1980), DeLong, Shleifer, Summers, and Waldmann (1990b), Arthur, Holland, LeBaron, Palmer, and Tayler (1997), Brock (1997), Brock and Hommes (1998), Gaunersdorfer (2000), Chiarella and He (2002b), Chiarella, Dieci, and Gardini (2002), Brock, Hommes, and Wagener (2005), Bottazzi, Dosi, and Rebesco (2005).

Since the return of the riskless asset r_f is not random, the solution of the problem in (3.2.1) reads:

$$x^*_{t,e} = \frac{1}{\beta\, W_t}\, \frac{\mathrm{E}[\rho_{t+1}] - r_f}{\mathrm{V}[\rho_{t+1}]} \quad . \tag{3.2.2}$$

Notation $x^*_{t,e}$ will become clear in the next Section. Thus, the *fraction* of the agent's wealth invested into the risky asset is proportional to the excess return of this asset and inversely proportional to the risk associated with the asset which is expressed as the variance of the return. These are rather reasonable properties of the demand. At the same time $x^*_{t,e}$ is inversely proportional to the agents wealth, which implies that investment into the risky asset decreases (in relative terms) with agent's wealth. The strangeness of such property can be further illustrated if one computes implied agent's demand for the risky asset:

$$D_{t,e} = \frac{x^*_{t,e}\, W_t}{P_t} = \frac{1}{\beta\, P_t}\, \frac{\mathrm{E}[\rho_{t+1}] - r_f}{\mathrm{V}[\rho_{t+1}]} \quad . \tag{3.2.3}$$

The current wealth has been disappeared from the right-hand side, so that agent demands the same amount of the risky asset independently of his initial wealth level W_t, other things being equal. Then the price equation does not contain the agents' wealths which leads to the lack of the wealth feedback loop in the models mentioned above.

The most straight-forward way to avoid such consequence of the demand specification is to define the demand function in, so to speak, pragmatic way. Instead of solution of some optimization problem like (3.2.1) for the demand derivation, one can start right away with individual demand functions and construct them taking into account all wishful features. This approach is, of course, subject to the critique from the viewpoint of the modern theory of decision-making under uncertainty. Nevertheless, it has been widely used in the agent-based models like in Day and Huang (1990), Chiarella (1992), Lux (1995, 1997), Sethi (1996), Franke and Sethi (1998) and Föllmer, Horst, and Kirman (2005).

Our framework can be, partially, seen as exploiting such pragmatism, but at the same time it can also be justified inside the expected utility maximization theory founded by John von Neumann and Oskar Morgenstern and outlined in their classical book von Neuman and Morgenstern (1944). Let us turn for the moment to such approach in order to describe another way of deriving demand (3.2.3). Then, we will be able to compare that function with other possible demand specifications.

3.2.2 Expected Utility Maximization

It is well known (and proven in, e.g. Grossman and Stiglitz (1980)) that if the agent expects the normal distribution of the risky asset's return ρ_{t+1}, then maximization of expected (conditional on the current agent's wealth) utility

$$\max \mathrm{E}[U(W_{t+1})] \qquad \text{s.t.} \quad W_{t+1} = W_t\,(1+r_f) + x_t\,W_t\,(\rho_{t+1} - r_f) \tag{3.2.4}$$

with so-called *negative exponential* utility function

$$U_e(W;\beta) = -e^{-\beta W}\ , \qquad \beta > 0 \tag{3.2.5}$$

is equivalent to the mean-variance optimization problem (3.2.1). Thus, the demand $D_{t,e}$ in (3.2.3) can be justified as the solution of expected utility maximization under one *additional assumption* concerning the agent's perception about return distribution.

Dependence of the demand on (i) utility function and (ii) expected return distribution is a common feature of the expected utility framework. However, the class of those demand functions which can be rigorously derived as solution of (3.2.4) under some reasonable continuous distributions of return is very limited. Apart from the case of exponential utility function (3.2.5), the demand can be derived explicitly for those investors who possess a *quadratic* utility function

$$U_q(W;a,b) = a\,W - b\,W^2\ , \qquad a,b > 0 \quad . \tag{3.2.6}$$

Even if no distributional assumptions on the asset return are needed in this case, the expected utility maximization with quadratic utility leads to even less realistic setting than exponential utility, since the resulting demand unboundedly decreases with increase of the wealth.

There is also third important class of utility functions. It generates the demand which increases with agent's wealth, as we would desire. This is a *power* utility function defined as follows:

$$U_p(W;\gamma) = \frac{W^{1-\gamma}-1}{1-\gamma}\ , \qquad \gamma > 0 \quad . \tag{3.2.7}$$

Positive coefficient γ describes the agent's attitude towards risk, but as we will see below its interpretation is slightly different from an interpretation of coefficient β in (3.2.1) and (3.2.5). When $\gamma = 1$ the expression in (3.2.7) is not defined but it can be extended continuously as *logarithmic* utility function

$$U_l(W) = \log W \quad . \tag{3.2.8}$$

It is obvious that expected utility maximization of the power (in particular, logarithmic) utility implies that the optimal fraction, which we call $x^*_{t,p}$, does not depend on the current wealth W_t. (The discussion of this property can be found in Mossin (1968), for example.) Therefore, the physical demand, expressed as

$$D_{t,p} = \frac{x^*_{t,p}\, W_t}{P_t} \quad , \tag{3.2.9}$$

increases with the agent's wealth. This property of the demand function is more realistic than the property of independence from the wealth which characterized demand $D_{t,e}$ in (3.2.3). It is not surprising, therefore, that classical finance literature has preferred to specify agents' behavior in the framework with power utility function (3.2.7), as it can be illustrated by the models in Samuelson (1969) and Merton (1969). As we mentioned in Section 3.1, this is not the case for the heterogeneous agent models, however. One of the main reasons for this phenomenon is, probably, the fact that, for the sake of deep analysis of the consequences of heterogeneity in expectations, these models tended to avoid an additional complexity arising when the wealth evolution has to be taken into account.

Another problem with the demand based on the expected utility maximization is the difficulty to derive *explicitly* the solution $x^*_{t,p}$ of optimization problem (3.2.4) with the power utility. The expression for $x^*_{t,p}$ is known for an analogous problem in continuous time (Merton, 1969), and in discrete time in the case when agent perceives the discrete distribution for the next period return. However, for all reasonable continuous distributions (e.g. log-normal) of the return the precise functional dependence of $x^*_{t,p}$ on the parameters of this distribution (analogous to (3.2.3)) is unknown. In Section 3.4 we will, first, discuss how this problem has been solved in the recent literature and, second, describe our own approach to this problem.

3.2.3 CRRA vs. CARA Frameworks

In order to complete the discussion about possible demand specification, let us introduce two measures of the agent's attitude towards the risk considered in the seminal works of Pratt (1964) and Arrow (1965). They define two following quantities:

Definition 3.2.1. For arbitrary utility function $U(W)$ an *absolute risk aversion* coefficient reads

$$\mathrm{ARA}(W) = -\frac{U''(W)}{U'(W)} \quad , \tag{3.2.10}$$

while a *relative risk aversion* coefficient reads

$$\mathrm{RRA}(W) = -\frac{U''(W)\, W}{U'(W)} \quad . \tag{3.2.11}$$

The negative signs in the two expressions above are due to the concavity of the utility function, so that both coefficients are positive for risk-averse investors. The second derivative of utility function measures its curvature. Both risk aversion coefficients are defined to be proportional to the curvature, since the (positive) difference between utility of the certain wealth and utility of the lottery around this wealth increases with curvature. The derivative $U'(W)$ in the denominators aims to eliminate any dependence on the units of utility. According to Pratt (1964), the absolute risk aversion coefficient (3.2.10) determines the absolute dollar amount that an investor is willing to pay to avoid a gamble of *absolute* small size. It is also shown that the relative risk aversion coefficient (3.2.11) determines the fraction of wealth that investor will pay to avoid a gamble of a given *relative* small size.

It is immediate to see that exponential utility function (3.2.5) is characterized by an independent of the current wealth absolute risk aversion coefficient β, while power utility function (3.2.7) is characterized by an independent of the current wealth relative risk aversion coefficient γ. Moreover, it is possible to show (see, e.g. Pratt (1964) for the formal derivation) that these are the only (to within the linear transformation) functions possessing these properties. This is the reason why the setting in which the demand function does not change with agent's wealth is referred as the Constant Absolute Risk Aversion (CARA) framework. Demand $D_{t,e}$ derived from an expected utility maximization with exponential utility function or from the mean-variance optimization in (3.2.1) is consistent with the CARA framework. The model analyzed in Chapter 2 was also built in CARA framework.

In contrast, the demand $D_{t,p}$ in (3.2.9) derived from expected utility maximization with power (or, in particular, logarithmic) utility is consistent with the Constant Relative Risk Aversion (CRRA) type of behavior. As we mentioned above, the optimal fraction $x^*_{t,p}$ in such demand function depends on the agent's perception about return distribution of the risky asset and does not depend on the current wealth W_t.

Even if $x^*_{t,p}$ (defined as a solution of an expected utility maximization with power utility) cannot be specified for continuous distributions, there is a way to get the demand in CRRA framework explicitly. In order to do it, one have to solve the following *mean-variance* problem

$$\max_{x_t} \left\{ \mathrm{E}[W_{t+1}] - \frac{\gamma}{2\,W_t}\,\mathrm{V}[W_{t+1}] \right\} \quad \text{s.t.} \quad W_{t+1} = W_t\,(1+r_f) + x_t\,W_t\,(\rho_{t+1} - r_f)\,. \tag{3.2.12}$$

Comparing this optimization problem with one given in (3.2.1), one can see that the coefficient of an absolute risk aversion β has been changed on coefficient γ/W_t. It implies that problem (3.2.12) has decreasing with wealth absolute risk aversion and constant relative risk aversion γ. The solution of (3.2.12) is provided by

$$x^*_{t,p} = \frac{1}{\gamma}\,\frac{\mathrm{E}[\rho_{t+1}] - r_f}{\mathrm{V}[\rho_{t+1}]}\quad , \tag{3.2.13}$$

and does not depend on the current wealth. Therefore, for the CRRA-version of the mean-variance optimization the demand function reads

$$D_t^{CRRA} = \frac{W_t}{\gamma\,P_t}\,\frac{\mathrm{E}[\rho_{t+1}] - r_f}{\mathrm{V}[\rho_{t+1}]}\quad , \tag{3.2.14}$$

which increases with the wealth of the agent, other things being equal. Notice that the derivation of the demand function (3.2.14) is performed without any specific assumptions about the distribution of return ρ_{t+1}.

The debates about the type of the agents' behavior prevailing in the markets based on the evidence from experimental economics are nicely summarized in Levy, Levy, and Solomon (2000). In particular, they discuss an experiment conducted in Kroll, Levy, and Rapoport (1988) who analyze the time-series data about 30 subjects repeatedly choosing the shares to be invested into the risky and riskless assets. This study rejects the hypothesis of the CARA type of behavior and in 15 cases also strongly supports the hypothesis of the CRRA type of behavior. In the remaining 15 cases an increasing relative risk aversion seems to prevail, but it may be a simple consequence of an asymmetry in the experiment design (subjects can win money but cannot lose out-of-pocket money). Levy (1994) conducted another experimental study in which wealth levels of the subjects were changing over time as a result of price evolution. The agents investment decisions affected the price dynamics as it also happens in the real markets. Also the asymmetry of the experiment design was eliminated. This experiment supported the hypothesis of constant relative risk aversion.

Our model will be build inside the CRRA framework. It is important to mention, however, that the discussion about "realistic" specification of the risk aversion are continuing in the experimental economics, and that other possibilities to incorporate the wealth feedback mechanism in the heterogeneous agent model are also possible. For instance, Holt and Laury (2002) conducted an experiment where each subject first participates in low-scale lottery decisions and then in large-scale lottery decisions. The comparison between the outcomes of these two lotteries leads to supporting the hypothesis of increasing relative risk aversion. Consequently, authors propose a hybrid, "power-expo" utility function that exhibits increasing CRRA and decreasing CARA coefficients for different wealth levels.

Let us sum up the discussion performed in this Section. We have shown that the most of the recent agent-based analytical contribution are built inside the CARA framework, where the feedback effect due to the wealth dynamics discussed in Section 3.1 is absent. Because of this reason, such framework is relatively simple to deal with, but, on the other hand, it leads to some unrealistic properties of the demand function. It is, therefore, not surprising that experimental literature tends to reject CARA-type of behavior in favor of the CRRA-type. In the next Section we present some recent examples of the modeling inside the CRRA framework.

3.3 Review of the Models with CRRA Agents' Behavior

We start this review with microscopic simulation approach summarized in Levy, Levy, and Solomon (2000). Then we discuss the literature in evolutionary finance started with a seminal paper Blume and Easley (1992).

3.3.1 Simulation Studies

Series of papers Levy, Levy, and Solomon (1994), Levy, Levy, and Solomon (1995), Levy, Persky, and Solomon (1996), Levy and Levy (1996), Levy and Solomon (1996), book Levy, Levy, and Solomon (2000), which appeared some years later, and the re-investigation of the same model in Zschischang and Lux (2001) provide an example of the simulation modeling of financial market. Being unconstrained by the necessity to deal with the model which possesses an analytical solution, authors build and analyze the setting with simple market structure but with relatively rich agents' behavior. The main question of all these contributions is the role of the heterogeneity in the trading behavior on the macro-dynamics.

The framework is the following. Market with one riskless and one risky asset is populated by finite number of agents. These agents are solving at each period expected utility maximization problem (3.2.4) with power utility (3.2.7), i.e. they share the CRRA type of behavior. The random term added to the optimal share $x^*_{t,p}$, describing an agent-specific demand component, introduces the first layer of heterogeneity in the model. Moreover, agents differ in the risk-aversion coefficient γ. The most important layer of heterogeneity in the behavior is introduced through the agents' expectations, however. Namely, agents expect that any of the last L returns can be realized with the equal probability and since the memory span L is assumed to be agent-specific, the agents' expectations are heterogeneous.

Authors are interested in the two aspects of their models. First, on the aggregate level, they look at the price and return trajectories and judge their "realisticity" without, however, careful time-series analysis. Second, on the basis of the evolution of the relative wealth of different agents, they analyze the determinants of the *dominance* of one agents' type over the other. The latter approach is common for the models built in CRRA framework as we will see later.

It turns out that the model generates unrealistically smooth time series in the homogeneous agents case. The same happens also in the case when the agent-specific noise is added to the agents' investment choices (Levy, Levy, and Solomon, 1994). Heterogeneity in the risk aversion coefficient leads to the dynamics with consequent booms and crushes which are predictable and, therefore, also unrealistic (Levy, Levy, and Solomon, 1995). Only an introduction of the heterogeneity in expectations (Levy, Persky, and Solomon, 1996) leads to the realistic time series. This is also the case if the agents are heterogeneous both in the expectations and in the attitude towards risk.

Analysis of the wealth evolution for different agents over time leads to the following results. First, population with larger memory spans L's tends to dominate the market. Second, investors with smaller risk aversion coefficients γ's tend to dominate the market. Moreover, as Zschischang and Lux (2001) claims

> Looking more systematically at the interplay of risk aversion and memory span, it seems to us that the former is the more relevant factor, as with different [risk aversion coefficients] we frequently found a reversal in the dominance pattern: groups which were fading away before became dominant when we reduced their degree of risk aversion.

Microscopic Simulation modeling leads to some important conclusions. In particular, it questions those classical contributions which are built under the assumption of representative agent stressing an importance of the heterogeneity between agents, and especially heterogeneity in expectations, for those phenomena which real financial markets display. On the other hand, it is not always clear what are the features responsible for generating certain results and how these results are robust with respect to the changes in the simulation setting. This weakness is common for all simulation models. In this particular case it may be illustrated by the phenomenon of the sensitivity of the resulting dominance pattern in the competition between different traders to the initial conditions (e.g. to the initial return history) discovered in Zschischang and Lux (2001).

3.3.2 Evolutionary Finance

Evolutionary Finance is an important branch of the analytical financial literature which considers the heterogeneous agents' behavior and takes into account the feedback mechanism based on the wealth dynamics. Seminal contribution of Lawrence Blume and David Easley (Blume and Easley, 1992) followed by other investigations in Blume and Easley (1993), Sandroni (2000), Hens and Schenk-Hoppé (2005b), Amir, Evstigneev, Hens, and Schenk-Hoppé (2005) and Sandroni (2005).

All these contributions consider the market as a big evolutionary system where different strategies[1] compete for the wealth. At the heart of this competition lies the wealth feedback effect described in Section 3.1 and implying that successful strategies possess greater power in the market outcome in the future. The market with finite number of traded assets selects the most "fit" among the set of heterogeneous strategies (portfolio rules) described as a vector of shares to be invested in different assets. Since these *investment shares* can be independent of current wealth, portfolio rules consistent with CRRA behavior are allowed.

The first question which should be addressed inside such framework concerns the derivation of the criterion according to which the notion of "fitness" can be defined. This question is answered in Blume and Easley (1992) who analyze the limit distribution of the wealth in the market and show that market "naturally selects" those traders who have the highest expected growth rate of the wealth shares. Having such criterion of survivance it is possible, as the next step, to address the question about validity of the Milton Friedman hypothesis which we discussed in Chapter 1. One can ask whether in the market "rational" strategies[2] always overperform irrational ones. As examples in Blume and Easley (1992, 1993) demonstrate, market can, indeed, drive out the irrational strategies. However, it happens only in some very particular cases of agents behavior, and, moreover, the class of rational strategies should be defined in rather restrictive way (as maximization of logarithmic utility (3.2.8)). In general, there are possibilities that rational traders will be driven out by other rational or even irrational traders. Similar findings concern the traders' expectations. There are cases when the traders with correct beliefs can be driven out of the market by others with incorrect beliefs. As a conclusion, "the link between market selection and rationality is weak".

Even if further investigation of Sandroni (2000) who consider complete markets and its generalization for incomplete markets in Sandroni (2005) show that the range of the agents' behavior among which Friedman hypothesis does apply can be enlarged, it is a general result of the evolutionary finance theory that

> ...there is nothing like "the best strategy" because the performance of any strategy will depend on all strategies that are in the market. Rationality therefore is to be seen as conditional on the market ecology.[3]

Judging the evolutionary finance literature it is important to mention that Hens and Schenk-Hoppé (2005b) and predecessors assume that rewards are drawn independently from time to time from exogenous distribution. In other words, they do not take into account an important feedback mechanism based on the return dynamics which presents in the real markets. Borrowing an example proposed in Blume and Easley (1992), one can say that evolutionary finance framework describes the betting on horse race but not the trade in financial markets where the capital gain constitutes an important part of the reward.

[1]In the models of evolutionary finance an importance of any individual agent in the market is defined exclusively by the total market wealth belonging to all the traders having the same investment behavior as this agent. Consequently, it is more reasonable to refer on the *strategies* and not on the *agents* competing in such a market.

[2]The question about meaning of the "rationality" is discussed in Blume and Easley (1992). In particular, any investment choice based on the expected utility maximization is included into the set of rational behaviors.

[3]See Hens and Schenk-Hoppé (2005a), p. 2.

Summing up the review of this Section we notice that the models from both branches of the literature which we have considered do not fit to the ideas outlined in Section 3.1. Simulation model does not allow strict investigation of generated market properties and (even more important) those determinants which are responsible for these properties. Evolutionary finance literature does not take into account capital gain term in the agents' reward and, therefore, misses one important return feedback loop.

The analytical model which we start to build in Section 3.5 will share the wealth feedback mechanism with evolutionary finance models. As a natural consequence, we will also be able to analyze the questions of strategies survival and market selection. However, all our results will be obtained in the framework with endogenous return determination which makes an important difference with evolutionary finance literature. At the same time, on methodological level we will borrow from evolutionary finance the idea to look at the limit behavior of economies *without* making hypotheses about individuals' motivations. Consequently, some implications which we derive for our model are very similar with those for evolutionary finance models. For example, in Chapter 4 we will find the same criterion for selection, the trader with highest growth rate of wealth survives. Also we show that the set of the strategies surviving in the market depends on the ecology of the strategies in the market, and, therefore, rational strategies are not necessary the "best" or "good". The absence in our model of the specification of the individual behavior leads to the fact that we are capable to incorporate those agents' behaviors which are considered in the Levy, Levy and Solomon model. Then, findings from the simulation studies can be explained and rigorously formalized.

3.4 Towards a Heterogeneous Agent Model in CRRA Framework

The last step which we have to undertake before starting the presentation of our model is to have a look on previous analytical models where both feedbacks are considered. To the best of our knowledge, there are only four such models. These contributions are discussed in the first part of this Section, while in the second part we, on the basis of this discussion, present the ideas which will underlie our own model.

3.4.1 Four Recent Contributions

First paper which we discuss here is due to Cabrales and Hoshi (1996). Both endogenous return determination and wealth evolution are assumed in the model which is different from all other contributions, because it is built in the continuous time. The setting is similar but simpler than one in Blume and Easley (1992). There are two assets, risky and riskless, and *two* agents (or two types of agents) maximizing power utility function of consumption. Both of these agents build their expectations on the basis of belief that return on the risky asset follows a geometrical random walk. Agents' expectations differ in the drift term, i.e. in the rate of return on the risky asset. "Optimistic" investor assumes a higher rate of return than "pessimist". Consequently, in terminology of Blume and Easley (1992), both investors have the constant investment strategies.

In the original model of Blume and Easley (1992), only one investor among two survives in the market with constant investment strategies. However, the model of Cabrales and Hoshi predicts that both investors can also coexist in the long-run. More precisely, depending on the values of parameters in the model, the random variable describing the relative wealth of the optimistic strategy can either converge with positive probability (sometimes almost sure) to 0, or converge with positive probability (sometimes almost sure) to 1. Such result is, clearly, a consequence of the relaxing of the assumption of exogeneity in the payoff specification. As we will see in the next Chapters, traders with constant investment strategies can coexist in the long run also in our model. Even if we obtain the same result, the traders' behavior leading to it is different in our model. Notice that the traders in Cabrales and Hoshi (1996) derive the demand from expectations about price return. The price evolving but expectations never change. Thus, in the case of coexistence in the market, both traders in this model make systematic mistakes, i.e. each of them is wrong in average. Our model does not require introduction of expectations, but even if they are introduced, the dynamics in equilibrium is always consistent with the expectations of surviving agents.

Contribution of Chiarella and He (2001) is a counterpart of simulation study of Levy, Levy, and Solomon (2000) in what concerns the market specification and agents' behaviors. The model is also very close to the traditional heterogeneous agent approach. It is an analytical, parsimoniously parameterized model in discrete time. Simple market with one risky and one riskless asset is populated by few types of agents who differ in their expectations about return for the risky asset. The agents are assumed to be the logarithmic expected utility maximizers[4] in choosing their optimal shares to invest into the risky asset. Since, as we mentioned above, the precise functional form of the demand cannot be derived for the CRRA agents in the discrete time setting for continuous return distribution, Chiarella and He provide an approximated demand function on the basis of the continuous time approximation. Such approach is also used in the further generalizations of the model (Chiarella and He, 2002a; Chiarella, Dieci, and Gardini, 2004), which opens the space for one possible critique of all these models since no mistakes of the approximation are provided.

With the specification of the demand function, agents can be divided into classes according to their expectations. There are *fundamentalists* who always invest constant shares of their wealth and also *chartists* who build their investment strategies on the basis of the return history. It is important to mention that all traders in the model make systematic mistakes in equilibrium, i.e. their expectations about return always differ from actual equilibrium return. Chiarella and He (2001) provide equilibria and stability analysis of their model for two cases, (i) the homogeneous expectations situation when all the agents are identical, and (ii) the heterogeneous expectations case when agents of two different types present in the market together. However, most of the stability properties derived in the paper are based on the numerical investigations, so that only the simplest case of the heterogeneous market without chartists is analyzed rigorously.

As a common property which connects all their findings, Chiarella and He (2001) formulate "quasi-optimal selection principle" according to which market always selects the (stable) equilibrium with the highest possible return. In Chapter 6, where we will reproduce this model inside our general framework, we will demonstrate that quasi-optimal selection principle does not have a universal character but, instead, is a consequence of those peculiar behaviors which Chiarella and He assume in their model.

Chiarella and He (2002a) consider the extension of the previous model by adding a switching mechanism among different types of trading strategies analogous to the one introduced in Brock and Hommes (1998). Now the strategies of the agents are not "frozen" but can be chosen according to the past performance based on the natural fitness measure, realized wealth. It allows to characterize various psychological effects like overconfidence or herd behavior by

[4]The generalization of the results on the case of power utility maximizers is straight-forward.

means of one additional parameter, intensity of switching between strategies. The extended model is no longer analytically treatable even in the simplest cases, however, so that almost all results are based on the numerical simulations.

Further step is undertaken in Chiarella, Dieci, and Gardini (2004), where the market structure is modeled in a more realistic way than in two previous models. Instead of assuming identical and independent distribution of the dividend yield, in the latter contribution authors consider the geometrical random walk for the dividend process. In addition, Walrasian framework for the price determination exploited in Chiarella and He (2001, 2002a) has been replaced by the market-maker scenario. Unfortunately, the resulting model turns out to be even more difficult to handle analytically. To simplify the matter, authors introduce some assumptions which cast doubts on the robustness of the results. For example, without any reasonable explanation authors assume the hyperbolic tangent function as the demand of chartists, which was initially supposed to be derived from the expected utility maximization problem. Even more crucial, they restrict the analysis on the case with zero outside supply of shares (the same assumption which has been introduced in Brock and Hommes (1998)), which can lead to completely different dynamics as we mentioned in Chapter 2. Finally, as in two previous contributions the stability analysis and global dynamics are studied by means of the computer simulations, since even such simplified model leads to the complicated dynamical system.

In this Section we discussed four contributions which incorporated those ideas on the basis of which we would like to build our model. All these contributions can be seen as different extensions of the models reviewed in Section 3.3. All four extensions are important for us, since they are made in the directions which are relevant from our point of view outlined in Section 3.1. However, apart of the problems with the agents' systematic mistakes and validity of the demand approximation, all these models share two important drawbacks. First, the market analyzed there is populated by *few* (one or two) classes of the agents. Thus, only small amount of heterogeneity is allowed. Second, the behavior of the agents is modeled always in very specific way. The point here is not that such behavior can turn out to be non-rational. The point is that *any* specification of the demand functions casts doubts on the generality of the results.

3.4.2 Alternative Approach Based on Investment Functions

Based on that considerations we suggest another way of dealing with the problem of the demand specification. Since the precise form of the investment share $x^*_{t,p}$ in the case of CRRA demand is unknown for the utility maximizers and also because evolutionary finance suggests not to limit the investigation on the rational strategies, it seems very reasonable *do not specify the investment shares at all.* This is exactly what is done in the model which we start to build in the next Section and will analyze in the remaining Chapters. That is, instead of deriving the agent's individual demand, we model, in total generality, the agent's investment choice as a smooth function of the available information. Such *investment function* can be agent-specific, partially due to the fact that its shape should somehow depend on the agent's attitude towards risk, and partially due to the different possible ways in which an agent can transform an available information set (public or private) into predictions about the future. In particular, the investment function can be derived from the rational approach either through the utility maximization or through the approximation. In the former case we, without precise knowledge of the investment function, substantially repeat the setting exploited in Levy, Levy, and Solomon (2000). In the latter case we can discuss explicit strategies used in Chiarella and

He (2001) as well as the strategies based on other approximations. In our investigation we will not confine the analysis to the case of two agents, but instead will deal with the situation in which a fixed, but arbitrary large, number of agents operate, at the same time, in the market.

Our model can be characterized as follows. It is a dynamical agent-based model of the pure exchange economy in which both price and wealth dynamics are endogenous, so that current price of the asset affects the rate of return of the asset bought in the previous period. The market structure is simple and, in particular, repeats the setting in Chiarella and He (2001). We consider a two-asset economy. The first asset is a riskless security, yielding a constant return on investment. This security is chosen as the numéraire of the economy. The second asset is a risky equity, paying a stochastic dividend. Trading takes place in discrete time and in each trading period the price of the risky asset is fixed by imposing market clearing condition on the aggregate demand function. Thus, we consider a consequence of the temporary market equilibria and analyze resulting dynamics.

Such economy is populated by a fixed number of agents acting as speculative traders. Agent participation to the market is described in terms of his individual demand for the risky asset. The individual demand can adapt accordingly with the information derived from the past market history. The individual demand functions are assumed proportional to traders' wealth. That is, at each time step, each trader expresses his demand for the asset, relative to a certain notional price, as the share of his present wealth he is willing to invest in that asset. This assumption is consistent with, but not limited to, the behavior based on an expected utility maximization, with a constant relative risk aversion utility function. Both for definiteness and for mathematical tractability we will focus our analysis on the case in which the agents investment functions depend solely on past aggregate market performances, that is we consider, so to speak, "technical trading" behaviors.

It is important to mention, that in our framework we do not distinguish between agents and investment strategy. This is so, because if two or more agents use the same strategy, then the relative performance of these agents will remain the same. At the same time, the performance of their common strategy will depend on the total wealth belonging to all the agents with such strategy. This is the same feature which one can find in the models of evolutionary finance. In the terminology of some heterogeneous agent models (Brock and Hommes, 1998; Brock, Hommes, and Wagener, 2005) we consider finite but arbitrary number of types of any, finite or infinite, number of traders.

Our goal will be to study the dynamical equilibria of the market[5], i.e. the asymptotic properties of the price and the wealth dynamics, when different heterogenous agents populate the economy. Performing this task in total generality concerning the number of agents and investment functions, we immediately extend some of the previous contributions among which analytical model of Chiarella and He (2001) and the microscopic model discussed in Section 3.3.1.

We are able to provide a complete characterization of market equilibria and a description of their stability conditions in terms of few parameters derived from traders' investment functions. It turns out that, irrespectively of the ecology of the agents operating in the market, the location of all steady-states can be illustrated by means of a simple function, the "Equilibrium Market Line" (EML). This separation between the underlying market structure which leads to the definition of the EML, and the agents' investment behaviors determining the precise

[5]We stress here once again that there are two meaining for the word "equilibrium". Here by "equilibrium" we understand the dynamical equilibrium, i.e. appropriately defined fixed point of the dynamical system describing the market evolution. Irrespectively of this notion, at each time period the price is defined in such a way that demand is equal to supply, i.e. market is in Walrasian temporary equilibrium.

equilibrium points, allows us to derive, notwithstanding the generality of the framework, several important conclusions. In particular, we find that, irrespectively of the number of agents operating in the market and of the structure of their demand functions, only two types of equilibria are possible: generic equilibria, associated with isolated fixed points, where a single agent asymptotically possesses the entire wealth of the economy and non generic equilibria, associated with continuous manifolds of fixed points, where many agents possess finite shares of the total wealth. This finding can be seen as a deterministic counterpart (inside our framework) of the corresponding results in Blume and Easley (1992).

Through the stability analysis we are able to describe the relative asymptotic performances of the different investment functions and, ultimately, the mechanism with which market selects the surviving traders. In this way the "quasi-optimal selection principle", originally formulated in Chiarella and He (2001) for (approximated) logarithmic utility maximizers, is made more accurate and extended to generic investment functions and arbitrarily large markets. This extension reveals the local nature of the market selection process and the impossibility to define any global dominance order relation among agents. Also these results have obvious connections with evolutionary finance literature.

Let us turn to the formal model now.

3.5 Model Setup

We consider a simple pure exchange economy, populated by a fixed number N of traders, where trading activities are supposed to take place in discrete time. The economy is composed by a risk-less asset (bond) giving in each period a constant interest rate $r_f > 0$ and a risky asset (equity) paying a random dividend D_t at the beginning of each period t. The risk-less asset is considered the numéraire of the economy and its price is fixed to 1. The ex-dividend price P_t of the risky asset is determined at each period, on the basis of its aggregate demand, through the market-clearing condition.

Suppose that this simple economy is populated by a fixed number of N traders. The dynamics through each period proceeds as follows. Before time t, after the end of previous trading round, agent n $(n = 1, \dots, N)$ possesses $A_{t-1,n}$ shares of the risky asset and $B_{t-1,n}$ shares of the risk-less asset. In the beginning of time t the agent gets (in terms of the numéraire) random dividends D_t per each share of the risky asset and constant interest rate r_f for all his riskless assets. Therefore, the wealth of the agent can be computed for any notional price P as

$$W_{t,n}(P) = A_{t-1,n}\, P + B_{t-1,n}\,(1 + r_f) + A_{t-1,n}\, D_t \quad . \tag{3.5.1}$$

Suppose that agent n decides to invest a fraction $x_{t,n}$ of his wealth into the risky asset and the remaining fraction $1 - x_{t,n}$ into the riskless asset. Thus, we consider the model without consumption where total wealth has to be reinvested. According to our timing specification, wealth fraction $x_{t,n}$, which represents a complete description of the agent behavior, is determined on the basis of the last dividend realization together with all information available from the previous periods, which we summarize in the information set $\mathcal{I}_{t-1}$. The individual demand for the risky asset becomes

$$\frac{x_{t,n}\, W_{t,n}(P)}{P} \quad ,$$

which is consistent with the CRRA type of behavior as soon as investment share $x_{t,n}$ is independent of the current wealth (cf. expression for CRRA demand in (3.2.9)). The aggregate

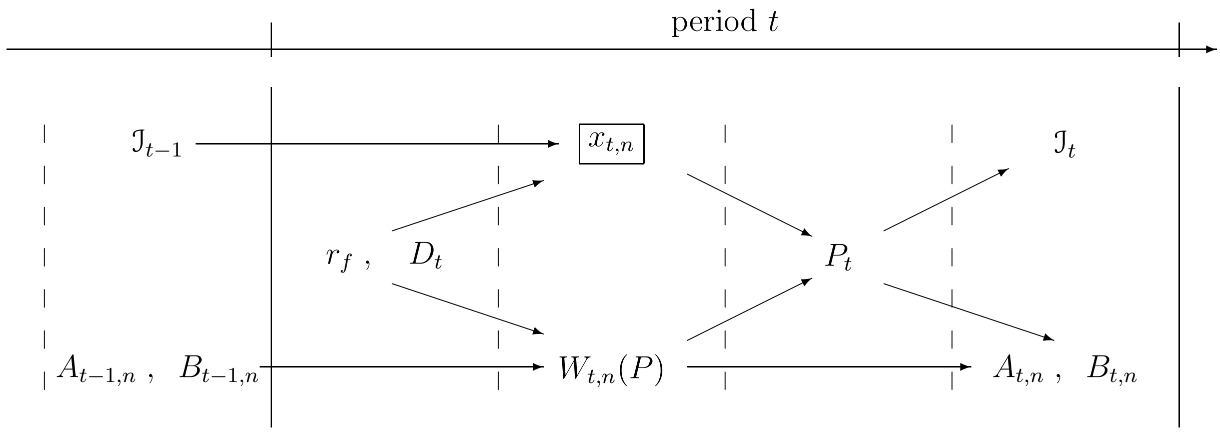

Figure 3.1: A diagram of the trading round at period t. In the beginning of period t the interest rate and dividend are paid. Then the trading round starts, in which the participation of any agent is limited to the choice of investment share $x_{t,n}$ (shown inside the box). This share depends, in general, on the information set $\mathfrak{I}_{t-1}$ available to the agent and also on the last dividend payoff. The investment share determines the individual demand which also depends, through the variable $W_{t,n}(P)$, on the current portfolio composition. Crucial assumption leading to CRRA framework is that $x_{t,n}$ does not depend on $W_{t,n}(P)$. With specified individual demand, the clearing price P_t can be fixed, and, at the same time, the new agents' positions in the assets are determined. The fixed price enhances the information set.

demand is the sum of all individual demands, and the realized price is defined as one where aggregate demand is equal to aggregate supply. Assuming a constant supply of risky asset, whose quantity can then be normalized to 1, the realized price P_t is determined as the solution of the equation

$$\sum_{n=1}^{N} x_{t,n}\, W_{t,n}(P) = P \quad . \tag{3.5.2}$$

At this moment also the new portfolio of any agent is determined. Namely, new positions of agent n in the risky and riskless assets read:

$$\begin{aligned} A_{t,n} &= \frac{x_{t,n}}{P_t}\big(A_{t-1,n}\, P_t + B_{t-1,n}\,(1+r_f) + A_{t-1,n}\, D_t\big) \quad , \\ B_{t,n} &= (1-x_{t,n})\big(A_{t-1,n}\, P_t + B_{t-1,n}\,(1+r_f) + A_{t-1,n}\, D_t\big) \quad . \end{aligned} \tag{3.5.3}$$

When the new portfolio of each agent is determined, the economy is ready for the next round. A diagram describing the different steps composing a trading session is presented in Fig. 3.1.

For our purposes the wealth dynamics will be more useful. Let $W_{t,n}$ denote the wealth of agent n during the trading session at time t. Notice that the first equation in (3.5.3) implies that $A_{t,n} = x_{t,n}\, W_{t,n}/P_t$, while the second equation leads to $B_{t,n} = (1 - x_{t,n})\, W_{t,n}$. The direct substitution of these two relations into (3.5.1) rewritten for time $t+1$ gives the following:

$$W_{t+1,n} = W_{t,n}\,(1 - x_{t,n})\,(1 + r_f) + \frac{W_{t,n}\,x_{t,n}}{P_t}\,(P_{t+1} + D_{t+1}) \quad . \tag{3.5.4}$$

The evolution of wealth, therefore, is described by two terms. The first term in the right-hand side reflects the contribution of the risk-free interest payment, while the second term contains the capital gain and the dividend payments.

Combining pricing equation (3.5.2) rewritten for time $t+1$ with wealth evolution (3.5.4) we get the following system

$$\begin{cases} P_{t+1} = \displaystyle\sum_{n=1}^{N} x_{t+1,n}\,W_{t+1,n} \\ W_{t+1,n} = W_{t,n}\,(1 - x_{t,n})\,(1 + r_f) + \dfrac{W_{t,n}\,x_{t,n}}{P_t}\,(P_{t+1} + D_{t+1}) \qquad \forall n \in \{1, \ldots, N\}\,. \end{cases} \tag{3.5.5}$$

This system describes the agents' wealth and market price evolution implying their simultaneous determination at each time step, as it also happens in the real markets. It will be useful to make some transformation of the variables in this system before performing a further analysis.

3.5.1 Rescaling the Economy

The dynamics described by (3.5.5) represents an economy which is intrinsically growing. In order to see that, let us sum the second equation in (3.5.5) over all the agents to obtain the dynamics of the total wealth

$$W_{t+1} = W_t\,(1 + r_f) + \big(P_{t+1} + D_{t+1} - P_t\,(1 + r_f)\big) \quad . \tag{3.5.6}$$

From (3.5.6) it is immediate to see that the presence of a constant positive riskless return r_f introduces an "exogenous" expansion of the economy, due to the continuous injections of new bonds, whose price remains, under the assumption of totally elastic supply, unchanged. This effect is obvious if one assumes that the market is perfectly efficient and no arbitrage is possible. Under the no-arbitrage hypothesis, indeed, the expected value at time t of the second term in the right-hand side of (3.5.6) has to be equal to zero, so that $\mathrm{E}_t[W_{t+1}] = W_t\,(1 + r_f)$. Consequently, the total wealth is characterized by an unbounded steady increase.

It is convenient to remove this exogenous economic expansion from the dynamics of the model. To this purpose we introduce the rescaled variables

$$w_{t,n} = \frac{W_{t,n}}{(1 + r_f)^t} \quad , \qquad p_t = \frac{P_t}{(1 + r_f)^t} \quad , \qquad e_t = \frac{D_t}{P_{t-1}\,(1 + r_f)} \quad , \tag{3.5.7}$$

denoted with lower case names. The last quantity e_t represents (to within the factor) the dividend yield. The relations between the different quantities expressed in terms of the unscaled, upper case, variables and the rescaled, lower case, ones can be easily obtained using (3.5.7). In particular the return from the risky asset R_{t+1} reads

$$R_{t+1} = \frac{P_{t+1} - P_t}{P_t} = r_{t+1}(1 + r_f) + r_f \quad , \tag{3.5.8}$$

where r_{t+1} denotes the price return in terms of the rescaled prices

$$r_{t+1} = \frac{p_{t+1}}{p_t} - 1 \quad .$$

Thus, zero total return for the rescaled prices $r_{t+1} = 0$ corresponds to the risk-free actual return $R_{t+1} = r_f$.

We rewrite the market dynamics defined in (3.5.5) using the set of variables introduced in (3.5.7) to obtain

$$\begin{cases} p_{t+1} = \sum_{n=1}^{N} x_{t+1,n}\, w_{t+1,n} \\ w_{t+1,n} = w_{t,n} + w_{t,n}\, x_{t,n} \left(\dfrac{p_{t+1}}{p_t} - 1 + e_{t+1} \right) \qquad \forall n \in \{1, \dots, N\} \quad . \end{cases} \tag{3.5.9}$$

In what follows we will analyze this system and work with rescaled variables. All our results can be easily traced back to the original system (3.5.5) using the definitions of the rescaled variables in (3.5.7) and transformation rules like (3.5.8).

As we mentioned above, our framework implies a simultaneous determination of the wealth and price. Indeed, the N variables $w_{t+1,n}$ defined in the second equation of (3.5.9) appear on the right-hand side of the first one, and, at the same time, the variable p_{t+1} defined in the first equation appears in the right-hand side of the second one. Due to that simultaneity, $N+1$ equations in (3.5.9) provide only an *implicit* definition of the state of the system at time t.

In order to derive *explicit* relations we can treat system (3.5.9) as an algebraic system with respect to price p_{t+1} and wealth $w_{t+1,n}$. Such consideration is correct, however, only if two important conditions are satisfied. First, the investment share $x_{t+1,n}$ for each agent should be independent of the agent's current wealth. Second, the investment shares $x_{t+1,n}$ should be also independent of the current price. As we discussed extensively above, the first requirement is consistent with the CRRA framework. Second requirement is not necessary satisfied in the classical CRRA framework, since the current price is traditionally included in the information set. However, the illustration in Fig. 3.1 suggests that in the dynamical setting it is rather natural to assume the contrary. Information available to the agent before the trade at time t contains past prices and dividends, but not current price.

Thus, from now on we will assume that condition of an *independence of investment choices* $x_{t,n}$ *from the current wealths and price* holds. This assumption, which we make explicit in Section 3.8, is crucial for the solution of system (3.5.9) with respect to the wealth and price, performing in the next Section.

3.6 Dynamical System for Wealth and Return

Transformation of the implicit dynamics of the price and wealth given by system (3.5.9) into an explicit one is not generally possible and entails restrictions on the possible market positions available to agents. This can be easily seen in the case of a single agent. Suppose that only one agent operates in the market, and suppose that he possesses B bonds and 1 equity (i.e. the total supply). If the agent decides to invest a share x of capital in the risky asset, the price equation would read

$$x\,(B + P_t) = P_t \quad . \tag{3.6.1}$$

The left- and the right-hand sides of this equations are linear functions of the asset price P_t and the equilibrium price is thus obtained by the intersection of these two straight lines. It is immediate to see that if $x = 1$ the two lines never intersect and if $x > 1$ the intersection is in the third quadrant and gives a negative price. We are then led to assume $x < 1$, i.e. to forbid short positions in bonds B. In the dynamical setting such requirement transforms into the restriction on possible *sequences* of investment shares, since the current trader position depends on his past investment. Hence, in the general N-agent case we expect to have a restriction on the possible values of the N-tuple $(x_1, \dots, x_N)$'s over time.

Let us introduce some notation useful to formulate the market dynamics in a more compact form. Let a_n be an agent specific variable, possibly dependent on time t. We denote with $\langle a \rangle_t$ its *wealth weighted average* at time t on the population of agents, i.e.

$$\langle a \rangle_t = \frac{\sum_{n=1}^{N} a_n \, w_{t,n}}{w_t} = \sum_{n=1}^{N} a_n \, \varphi_{t,n} \quad , \quad \text{where} \qquad w_t = \sum_{n=1}^{N} w_{t,n} \tag{3.6.2}$$

represents the total wealth in the economy and $\varphi_{t,n} = w_{t,n}/w_t$ is the n-th agent's wealth share.

The next result gives the condition for which the dynamical system implicitly defined in (3.5.9) generates economically meaningful dynamics, i.e. dynamics where the asset prices remain positive, and also provides an explicit description for such dynamics

Proposition 3.6.1. *From system* (3.5.9) *it is possible to derive a map* $\mathbb{R}_+^N \to \mathbb{R}_+^N$ *that describes the evolution of wealth* $w_{t,n}$ $\forall n \in \{1, \dots, N\}$ *with positive prices* $p_t \in \mathbb{R}_+$ $\forall t$ *provided that*

$$\Big(\langle x_t \rangle_t - \langle x_t \, x_{t+1} \rangle_t \Big) \Big(\langle x_{t+1} \rangle_t - (1 - e_{t+1}) \langle x_t \, x_{t+1} \rangle_t \Big) > 0 \qquad \forall t \quad . \tag{3.6.3}$$

If this is the case, the growth rate of (rescaled) price $r_{t+1} = p_{t+1}/p_t - 1$ *reads*

$$r_{t+1} = \frac{\langle x_{t+1} - x_t \rangle_t + e_{t+1} \, \langle x_t \, x_{t+1} \rangle_t}{\langle x_t \, (1 - x_{t+1}) \rangle_t} \quad , \tag{3.6.4}$$

the individual growth rates of (rescaled) wealth $\rho_{t+1,n} = w_{t+1,n}/w_{t,n} - 1$ *are given by*

$$\rho_{t+1,n} = x_{t,n} \left(r_{t+1} + e_{t+1} \right) \qquad \forall n \in \{1, \dots, N\} \quad , \tag{3.6.5}$$

and the agents' (rescaled) wealth shares $\varphi_{t,n}$ *evolve accordingly to*

$$\varphi_{t+1,n} = \varphi_{t,n} \, \frac{1 + (r_{t+1} + e_{t+1}) \, x_{t,n}}{1 + (r_{t+1} + e_{t+1}) \, \langle x_t \rangle_t} \qquad \forall n \in \{1, \dots, N\} \quad . \tag{3.6.6}$$

Proof. See appendix C.1. □

The market evolution is explicitly described by the system of $N + 1$ equations in (3.6.4) and (3.6.5), or, equivalently, in (3.6.4) and (3.6.6). The price dynamics can be derived from the return evolution (3.6.4) straight-forwardly, but the price will stay positive only as long as condition (3.6.3) is satisfied at any time step.

The value of the return derived in (3.6.4) depends on both components of the demand: the agents' investment shares and agents' wealth. Concerning the latter, we notice that the *relative* wealth matters for the return determination. It justifies our intuition that relatively wealthy agents have higher influence in the market. Concerning the former we find that the shares from *two* periods are relevant as expected in the framework with endogenous price determination. High value of the agent's investment choice increases current total demand for the risky asset. Then current price and return go up. Consequently, the return is an increasing function of the contemporaneous investment shares. One may expect the opposite relation with respect to the choices made in the previous period, since high price at time t implies small return at time $t+1$. Expression in (3.6.4) confirms this intuition only partially. Without dividend payment, the current return is, indeed, the decreasing function of the investment shares from previous period. However, this relation may be reverted in the presence of the dividend payments. The reason is simple. The dividends paid in the beginning of time $t+1$ are proportional to the demand for the risky asset in period t. At the same time, the dividends, through the agents' wealth, contribute to the increase in the current demand and price. It creates an opposite effect of the influence of the previous period demand on the current return.

Expression in (3.6.5) shows that individual return is determined by the gross return, which includes the capital gain or loss and the dividend yield. Any agent gets part of this gross return proportional to the share which he had invested. Thus, investor having initially small impact in the growing market can earn a high profit investing high share $x_{t,n}$ in the risky asset. There is a risk of high losses, however, in the case of the negative gross return in the next period[6].

Equation (3.6.6) describes the evolution of the relative wealth. One can interpret this relation as a *replicator dynamics*, initially used in mathematical biology and then in evolutionary economics. Indeed, rewriting this equation for agent n as follows

$$\varphi_{t+1,n} = \varphi_{t,n} \frac{1+\rho_{t+1,n}}{1+\langle \rho_{t+1} \rangle_t} \quad , \tag{3.6.7}$$

allows us to see that the influence of the agent in the market changes according to his performance *relative* to the average performance, where one has to take the (rescaled) wealth return as a measure of performance.

Finally, let us discuss restriction (3.6.3). This condition is nothing else than an intertemporal many-agent generalization of the constraint derived from (3.6.1) on the permitted values of one agent's investment shares. If the inequality is not satisfied at some point in time, then positive price balancing demand and supply cannot be defined in the market at this moment. Otherwise, the price exists and unique. Thus, condition (3.6.3) gives *necessary and sufficient condition* for the existence of the sequence of temporary market equilibria in our framework. This constraint cannot be simplified even in the case when all agents have constant investment shares. The point is that any term in (3.6.3) is weighted with wealth whose distribution can change over time.

However, there exist a situation when this constraint is fulfilled at each time step. This case has simple economic meaning. Roughly speaking, one has to forbid the agents to go short both in the riskless and in the risky asset. Precise conditions are provided by the following

Proposition 3.6.2. *Consider the system defined in Proposition 3.6.1. If there exist two real values x_{min} and x_{max} such that*

$$0 < x_{min} \le x_{t,n} \le x_{max} < 1 \qquad \forall t \quad \forall n \in \{1, \ldots, N\} \quad , \tag{3.6.8}$$

and $e_t \ge 0$, then condition (3.6.3) *is always satisfied. Moreover, in this case, there exist constants r_{min}, r_{max}, ρ_{min} and ρ_{max}, such that for all time t it is:*

$$r_{min} \le r_t \le r_{max} \quad \forall t \qquad \textit{and} \qquad \rho_{min} \le \rho_{t,n} \le \rho_{max} \quad \forall t \quad \forall n \in \{1, \ldots, N\} \quad ,$$

so that the dynamics of (3.6.4) *and* (3.6.6) *is bounded.*

[6] Remember that we are dealing with rescaled returns, so that positive (negative) return corresponds to the unscaled return which is greater (lower) than r_f.

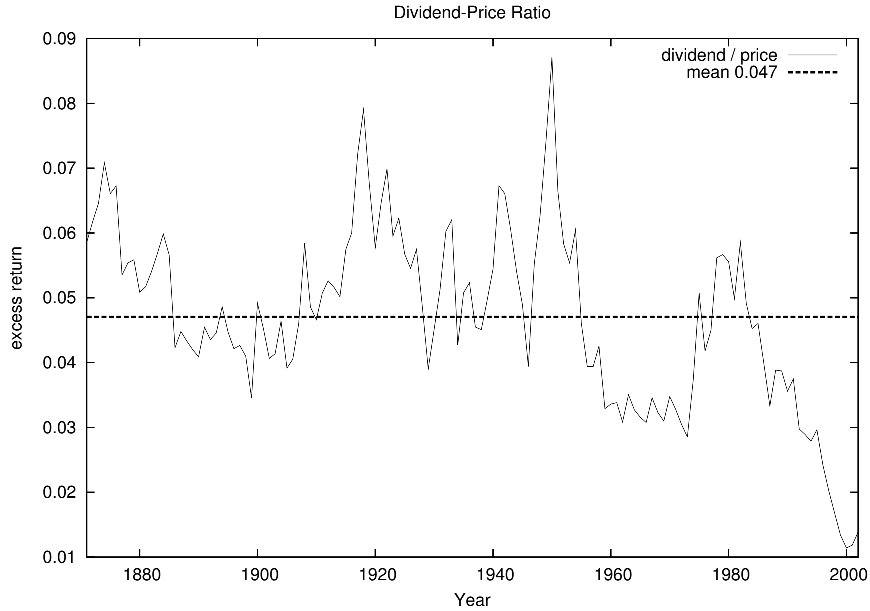

Figure 3.2: Dividend-price ratio. Source: data set on Standard&Poor 500 index by Robert Shiller available online `http://www.econ.yale.edu/~shiller/data.htm`.

Proof. See appendix C.2. □

Thus, if all possible investment choices are confined within some compact subinterval in $(0, 1)$, then equations (3.6.4) and (3.6.6) give a well-defined dynamics in terms of price and wealth shares. In the opposite case, when the agents are allowed to go short either in the riskless or in the risky asset, the existence of positive price is not guaranteed.

In order to close the system derived in Proposition 3.6.1, one has to provide stochastic (due to random dividend payment D_t) yield process $\{e_t\}$ and specify the set of investment shares $\{x_{t,n}\}$. This is done in the next two Sections.

3.7 Yield Dynamics

Since our main attention will be paid to the impact of the agents behavior on the market dynamics, we choose the simplest possible characterization of the exogenous dividend process. Notice that apart from the dividend yield term e_{t+1}, the system of equations (3.6.4) and (3.6.6) describes the dynamics is terms of the sole price and wealth returns. Then, it is convenient to eliminate possible dependence of the dividend yield on price. In order to do it, let us introduce the following

Assumption 1. The dividend yields e_t are i.i.d. random variables obtained from a common distribution with positive support, mean value $\bar{e}$ and variance σ_e^2.

This assumption implies that price and dividends grow at the same rate. Simple derivation shows that this property characterizes the fundamental price under the assumption of the geometrical random walk of the dividend yield. Notice however, that in our model the price is determined through the market clearing condition and, therefore, it is not necessary fixed on the fundamental level. Thus, Assumption 1 is more restrictive than it may seem. On the other hand, the annual historical data for the Standard&Poor 500 index which we report in Fig. 3.2 supports our assumption, suggesting that yield can be reasonably described as a bounded positive random variable whose behavior is roughly stationary. Moreover, Assumption 1 is common to several works in literature (Chiarella and He, 2001, 2002b). Consequently, a further reason to have this assumption introduced is to maintain comparability with previous investigations.

3.8 Agents Behavior: Investment Functions

According to our discussion in Section 3.4.2 we will not rely on any specification of the functional form for the demand function. Instead, we define it as general as possible inside our framework. In the next two Chapters, where we present the general analysis of the long-run dynamics, we will make this general specification slightly more specific.

One of the principle leading to our definition of the investment decisions is that these decisions, being idiosyncratic and endogenous, have to be independent of the contemporaneous price and wealth levels. Moreover, since in this work we are mainly concerned with the effect of *speculative* behaviors on the market aggregate performance, we let aside those issues which may occur under asymmetric knowledge of the underlying fundamental process. Thus, we assume that the structure of the yield process defined in Assumption 1 is known to everybody. Along the same line, we assume that all agents base their investment decisions at time t exclusively on the public and commonly available information set $\mathcal{I}_{t-1}$ formed by past realized prices. This set can alternatively be defined through the past return realizations as follows

$$\mathcal{I}_{t-1} = \{r_{t-1}, r_{t-2}, \dots\} \quad . \tag{3.8.1}$$

Past realizations of the fundamental process do not affect agents' decisions, which, instead, adapt to observed price fluctuations. One can refer to this investment behavior, common in the agent-based literature (e.g. Brock and Hommes (1998)), as *"technical trading"*, stressing the similarity with trading practices observed in real markets.

We make the following

Assumption 2. For each agent n there exists a differentiable **investment function** f_n which maps the present information set into his investment share:

$$x_{t,n} = f_n(\mathcal{I}_{t-1}) \quad . \tag{3.8.2}$$

Function f_n in the right-hand side of (3.8.2) gives a complete description of the investment decision of the n-th agent. The knowledge about the fundamental process, being complete and time invariant, is not explicitly inserted in the information set, rather is considered embedded in the functional form of f_n.

Our specification of the investment behavior is relatively rich. As we said above it includes, as a particular case, those demand functions which are derived as a solution of an expected utility maximization problem with power utility (implying agent's CRRA behavior). All such solutions corresponding different beliefs about future return distribution provide different investment functions. Our specification also includes solutions of mean-variance optimization problem (3.2.12) for different expectations about first two moments of future wealth. At the same time, Assumption 2 rules out the CARA type of behavior and also other possible dependences in the determination of portfolio composition of agents, like an explicit relation of the present investment choice with the past investment choices or with investment choices of other traders.

3.9 Conclusion

Based on the analysis of previous contributions, in this Chapter we have built an analytical model where both price and wealth are determined endogenously. We have derived the evolution of price and wealth in the market whose structure is described by Assumption 1 and which is populated by the agents behaving consistently with Assumption 2.

The next step consists in the characterization of an asymptotic market dynamics. Notice in this respect that our market structure is defined in such a way that dynamics in (3.6.4) and (3.6.6) does not depend on the price level directly, but, instead, is defined in terms of price return and dividend yield. Since in these equations only the wealth ratios defined in (3.6.2) appear, which are insensitive to a homogeneous rescaling of the wealth levels, the equilibria of the model can be identified as states of steady expansion (or contraction) of the economy. This is the reason why we prefer the specification (3.8.1) for the information set and not the equivalent definition of this set through the price levels. It is clear that the set of equations (3.6.4), (3.6.6) and (3.8.2) defines the stochastic dynamical system. In general, this system is infinitely dimensional due to the dependence of investment functions f_n on the entire history of realized returns.

The next Chapters will be devoted to the analysis of this system. First, in Chapter 4 we analyze the case when investment functions depend on the specific estimators summarizing all past information. The dimension of the system can be reduced due to the recursive property of these estimators. Then, in Chapter 5, we consider the case of generic investment functions depending on the finite number of past returns. In Chapter 6 we apply the previous analysis to a particular class of agents' behavior, roughly speaking, to the case of linear investment functions. We rigorously show that model in Chiarella and He (2001) is a particular case of our framework. Finally, in Chapter 7 we analyze our framework under noisy specification of the agents' behavior. We found that when the number of the "noisy" agents is infinitely large and some conditions on their investment choices are satisfied, then so-called "Large Market Limit" exists, in which the market dynamics is described as if one, "representative" agent presents in the market. Otherwise, the agent specific noise can have a great impact on the market and the standard representative agent specification is not valid.

Chapter 4

Individual Behavior Based upon Return Forecast

In this Chapter we solve the general model introduced in Chapter 3 for the case when individual behavior is based upon the specific estimators, playing the role of the agents' expectations. Even if we fix the set of estimators used by agents, heterogeneous expectations are allowed and explicitly introduced. It links this version of the general model with the literature on the heterogeneous agent modeling, where the agents expectations are considered as the most important source of heterogeneity. An alternative specification of our basic model, which we analyze in Chapter 5, will be more close to the evolutionary finance literature, instead, since it will assume the general heterogeneity in the behavior without any specification of its origins.

4.1 Investment Decisions as Two Step Procedure

The majority of the agent-based analytical models concentrate on the consequences of the heterogeneity in agents expectations. Often the model is built in such a way that the same demand function is assumed for all agents. This generic demand depends on the expectations about future reward and risk. When the assumption of heterogeneity of expectations is introduced in such framework, the realized demands turn out to be different. Such approach, for instance, has been used in DeLong, Shleifer, Summers, and Waldmann (1990b), Brock and Hommes (1998), Chiarella and He (2001). Only further generalizations of these models analyze the role of another layer of heterogeneity, e.g. through consideration of the agent-specific risk aversion degree like in Chiarella and He (2002b).

Direct application of this approach to our framework implies that the investment behavior of the agent, modeled in Section 3.8 by means of the general investment functions, has to be represented as a result of two distinct steps. In the first step agent n, using a set of estimators $\{g_{1,n}, g_{2,n}, \dots\}$, forms his expectation about the behavior of future returns

$$\mathrm{E}_n[\theta_j(r_{t+1})] = g_{j,n}(\mathfrak{I}_{t-1}) \ , \qquad j = 1, 2, \dots \quad ,$$

where θ_j stands for some characteristic of the return distribution at time $t+1$, like mean or variance of the return. With these expectations, using a choice function h_n, possibly derived from some optimization procedure, agent computes in the second step the fraction of the wealth invested in the risky asset so that $x_{t+1,n} = h_n(\mathrm{E}_n[\theta_1], \mathrm{E}_n[\theta_2], \dots)$. Under this specification of behavior, the investment function f_n introduced in Assumption 2 of Section 3.8 can be considered as a result of the composition of estimators $\{g_{\cdot,n}\}$ and choice function h_n.

In this Chapter we will analyze the version of the general model with such both intuitive and widespread specification.

Remember that in the general setting individual investment decisions are based on the entire past history of the market, which leads to an infinite-dimensional dynamical system. Two step procedure described above can help to overcome this difficulty, if the estimators $g_{\cdot,n}$ applied in the first step admit a *recursive* definition. In this Chapter we consider the case of exponentially weighted moving average (EWMA) estimators which we already used in the model of Chapter 2 in the case of CARA type of agents' behavior.

The rest of this Chapter is organized as follows. In the next Section we present a complete list of those assumptions which we use for the solution of the current version of the general model. In particular, we introduce the EWMA estimators and correspondingly modify definitions of the investment functions. In order to provide an easier reading, the discussion of the results is organized in successive steps, of increasing difficulty. In Section 4.3 we consider the simple case of homogeneous investment choices, what corresponds to single agent operating in the market, i.e. $N = 1$. We begin with the analysis of investment function that depends only on the previous period return. The possible equilibria and their stability conditions are derived and shortly discussed. In doing this we show that a simple function, the "Equilibrium Market Line", can be used to obtain a geometric characterization of the location of equilibria. Then, we generalize our findings to the case in which the homogeneous investment function depends on the general EWMA estimator of the average return. We conclude the discussion of the single agent case by another generalization in which the investment function depends also on the expected variance of future return. It turns out that accounting for the endogenous component of risk generated by the price volatility does not play any role in the determination of equilibria and in their local stability. The general case of heterogeneous expectations with many distinct agents operating in the market is analyzed in Section 4.4. We derive, under generic agents preferences, the system describing the evolution of the economy, characterize all the possible equilibria and study their stability. In Section 4.5 we present two applications of the obtained results. First, we show that the geometric tool of the Equilibrium Market Line can be useful in the qualitative analysis of the equilibria and their stability even for general, many agent markets. Second, we consider a simple family of individual investment functions and discuss the effect of the different parameters on the location and stability of market equilibria. Section 4.6 summarizes our findings.

4.2 Agents Investment Functions and EWMA Estimators

Our starting point in this Chapter is the asset-pricing model introduced in Section 3.5. In Proposition 3.6.1 we have derived the dynamics generated by this model for an arbitrary dividend process and general investment behavior. In this Chapter we leave the same specification of the dividend structure as in Section 3.7 reintroducing, therefore,

Assumption 1. The dividend yields e_t are i.i.d. random variables obtained from a common distribution with positive support, mean value $\bar{e}$ and variance σ_e^2.

At the same time we relax the generality of the agents' behavior introduced in Chapter 3 through Assumption 2 and substitute it by the following

Assumption 2′. For each agent n there exists a parameter $\lambda_n \in [0, 1)$ so that the agent's investment share can be obtain by means of a deterministic smooth **investment function** f_n as

$$x_{t,n} = f_n\big(y_{t,n}, z_{t,n}\big) \quad , \tag{4.2.1}$$

where $y_{t,n}$ and $z_{t,n}$ are the expectations about future price return and variance obtained with EWMA estimators from information set $\mathfrak{I}_{t-1}$, i.e.

$$\begin{aligned} y_{t,n} &= (1-\lambda_n) \sum\nolimits_{\tau=0}^{\infty} \lambda_n^{\tau}\, r_{t-\tau} \quad , \\ z_{t,n} &= (1-\lambda_n) \sum\nolimits_{\tau=0}^{\infty} \lambda_n^{\tau} \big(r_{t-\tau} - y_{t,n}\big)^2 \quad . \end{aligned} \tag{4.2.2}$$

Thus, in the current version of the model the investment decision of agent n is based upon two estimators describing the expected return and expected risk coming from the capital gain. The definition of the investment share of agent n is completed through function f_n in the right-hand side of (4.2.1), which maps all possible combinations of expected return and risk into the investment choice. Thus, function f_n is defined on the set $[-1, +\infty) \times [0, +\infty)$.

Remember from our discussion in Chapter 2 that parameter λ_n is a sort of "memory" parameter that determines how the relative weights in the averages (4.2.2) are distributed among recent and old observations. The weights are declining geometrically in the past, so that the last available observation r_t has the highest weight. The value $\lambda = 0$ corresponds to the case of *naïve* forecast, i.e. to the agent who uses the last realized return as a predictor for the next period return. The use of the EWMA estimators seems reasonable in a dynamical setting where agents take into consideration the possibility that the "mood" prevailing in the market may change over time, so that more recent values of the price return could contain more information about future prices than the older ones. Estimators (4.2.2) admit the following recursive definition

$$\begin{aligned} y_{t,n} &= \lambda_n\, y_{t-1,n} + (1-\lambda_n)\, r_t \quad , \\ z_{t,n} &= \lambda_n\, z_{t-1,n} + \lambda_n\, (1-\lambda_n) \big(r_t - y_{t-1,n}\big)^2 \quad , \end{aligned} \tag{4.2.3}$$

which we can use in order to reduce the dimension of the dynamical system governing the market evolution.

The dynamical system which we will analyze in this Chapter is composed by the set of equations describing the return dynamics (3.6.4), the relative wealth evolution (3.6.6), the agents investment behavior (4.2.1) and, finally, the evolution of the estimators (4.2.3).

In the previous Chapter we have established the necessary and sufficient condition for the existence of positive price in such framework. Namely, it exists when the inequality in (3.6.3) is satisfied at each time step. This is an important restriction which we will partially ignore in what follows. The point is that our analysis is concentrated on the long-run market characterization of the deterministic version of the system. Thus, we explicitly require the existence of positive price only in the eventual rest-point of the system, ignoring this issue along the transitional path and in the noisy version of the system. On the other hand, the result of Proposition 3.6.2 guarantees that unique positive price exists in the market with investment functions having compact ranges inside the interval $(0, 1)$.

4.3 Economy with Homogeneous Agents

In this Section we perform the equilibrium and stability analysis of the dynamics derived above for the simplest situation in which traders possess homogeneous beliefs and preferences, in other terms, share the same investment function. It is clear from Proposition 3.6.1 that the dynamics of the economy is equivalent to the one obtained when a single agent operates in the market, i.e. when $N = 1$. We will refer to this case as the "single agent" case and consider it at length, because of its relevance for the heterogeneous agents case. In the case of one single agent the dynamical system describing the market evolution can be considerably simplified since the explicit evolution of the wealth shares in (3.6.6) is not needed.

We start with the analysis of a single agent whose investment function[1] f only depends on the last return realization. We derive market equilibria and study their stability conditions. Subsequently, we extend the analysis to the case when an investment function depends on the general EWMA prediction about future price return y. After studying this extension, we turn to the general case when investment function also depends on the estimated variance z, as in (4.2.1). Splitting the investigation in different steps helps in understanding the different effects that past return, y and z have on the market dynamics. Before starting our analysis, it is useful to introduce the following

Definition 4.3.1. The *Equilibrium Market Line* (EML) is the function $l(r)$ defined according to

$$l(r) = \frac{r}{\bar{e} + r} \quad , \tag{4.3.1}$$

where $\bar{e}$ stands for the mean yield value as defined in Assumption 1.

This function l will play the main role in a geometric characterization of the equilibria of the models developed here and in the next Chapters.

4.3.1 Naïve forecast

The simplest possible case inside behavioral class defined in Assumption $2'$ consists in taking the last realized return as a predictor for the next period return. As we mentioned above this "naïve forecast" behavior corresponds to the case $\lambda = 0$. If one assumes that single agent conforms to this rule, dynamical system describing market evolution reduces to

$$\begin{cases} x_{t+1} = f(r_t) \\ r_{t+1} = \dfrac{f(r_t) - x_t + e_{t+1}\, x_t\, f(r_t)}{x_t\,(1 - f(r_t))} \end{cases} . \tag{4.3.2}$$

where function f is defined on $[-1, +\infty)$, i.e. for all possible price returns.

The stochastic nature of (4.3.2) originates from the random dividend yield $\{e_t\}$. In order to study the asymptotic properties of the system, we substitute the realizations of the yield process by its mean value $\bar{e}$ and consider the fixed points of the resulting *deterministic* skeleton. The following result characterizes their existence and location.

Proposition 4.3.1. *Let* (x^*, r^*) *be a fixed point of the deterministic skeleton of system* (4.3.2). *Then it is:*

[1] Since only one agent is present in the market, we omit index 1 from any agent-specific variable.

(i) The equilibrium return r^* and the equilibrium investment share x^* satisfy

$$l(r^*) = f(r^*) , \qquad x^* = f(r^*) \quad . \tag{4.3.3}$$

(ii) The equilibrium is feasible, i.e. the equilibrium prices are positive, if either $x^* < 1$ or $x^* \geq 1/(1-\bar{e})$.

(iii) The equilibrium growth rate of agent's wealth is equal to price return, $\rho^* = r^*$.

Proof. See appendix D.1. □

The equilibria of the dynamical system (4.3.2) are characterized according to Proposition 4.3.1*(i)*. Equations (4.3.3) provide a simple geometric way to find all possible equilibria: they can be obtained as the intersections of investment function f with the Equilibrium Market Line l defined in (4.3.1). As an example, consider the case of two investment functions drawn as thick lines in Fig. 4.1. The thin line represents the hyperbolic curve of the EML. This line is made of two branches separated by a vertical asymptote in $-\bar{e}$. As we said, all equilibria are identified as the intersections of investment functions with the EML. The abscissa of the intersection gives the value of the equilibrium return r^*, while the ordinate gives the equilibrium investment share x^*. In our example, the nonlinear function has two equilibria: S_1 with small positive (rescaled) return and U_1 with high positive (rescaled) return. The linear function also has two equilibria, S_2 with negative (rescaled) return and U_2 with high positive (rescaled) return.

Proposition 4.3.1*(ii)* shows that not all equilibria are economically meaningful, though. The equilibrium return generates positive prices only in those equilibria where $r^* \geq -1$, or, equivalently, if the investment share belongs to the intervals $(-\infty, 1)$ or $[1/(1-\bar{e}), +\infty)$. This condition is, indeed, the equilibrium version of inequality (3.6.3). The point E on the EML depicted in Fig. 4.1 separates all feasible equilibria from unfeasible. Finally, Proposition 4.3.3*(iii)* states that the growth rate of the agent's wealth coincides with the price return. This result is expected in the single agent economy. However, it is interesting that the interrelation between the total return $r^* + \bar{e}$ and the investment in equilibrium x^* is such that the total wealth grows with a rate which does not directly depend on the dividend yield.

The geometric plot of the EML allows to illustrate some important properties of our framework. For instance, it can be immediately seen that there are no equilibria with $r^* = -\bar{e}$. Indeed, if the negative price return exactly offsets the positive dividend yield, then the (rescaled) wealth of the agent is constant over time. But since the asset price decreases, the investment share should also steadily decrease, and, therefore, there exist no equilibria. This case, when return is equal to $-\bar{e}$, plays a special role in our model, since it corresponds to the situation in which two assets, the riskless and risky, are equivalent from the point of view of the expected return. In this sense, in any equilibria with $r^* \neq -\bar{e}$ one of the assets is preferable to another one. Consistently with the behavior which we assume for our investors, arbitrage opportunities arising in such equilibria are not exploited by any agent. However, as we will show in Section 4.4, when many agents present in the market, ”no arbitrage” equilibria with $r^* = -\bar{e}$ can exist due to the possibility for different agents to take opposite positions in the risky asset.

Fig. 4.1 allows to identify three qualitatively different equilibrium scenarios. In equilibria with $r^* \in [-1, -\bar{e})$ the investment in the risky asset is characterized by negative gross return $r^* + \bar{e} < 0$. In these equilibria the agent maintains a long position in the risky asset ($x^* > 1$)

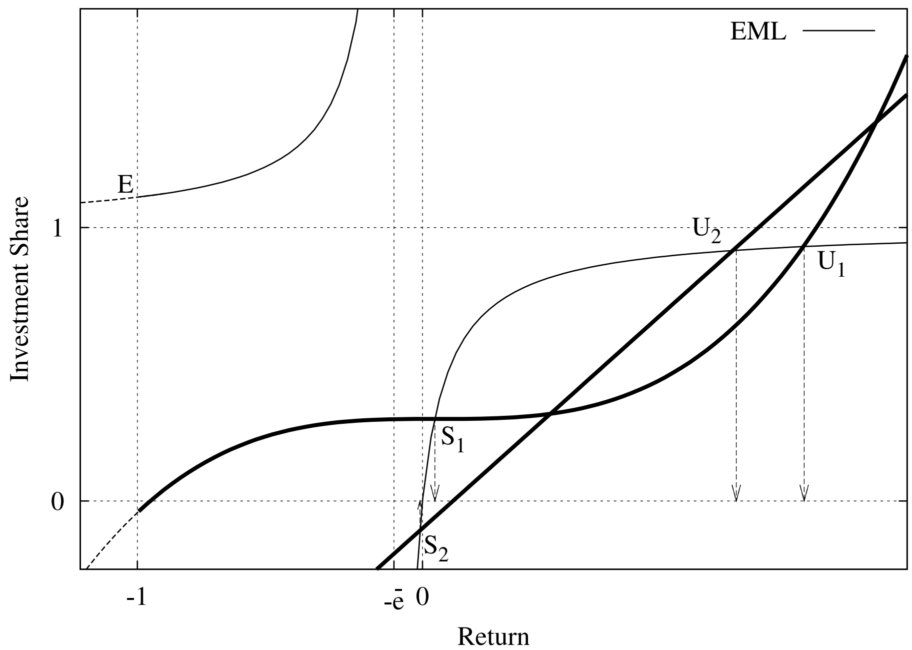

Figure 4.1: Equilibria for the single agent system for two different investment functions (thick lines) characterized as intersection with the EML (thin line).

and his wealth return is negative[2]. If $r^* \in (-\bar{e}, 0)$ the capital gain on the risky asset is negative, nevertheless the gross return is positive due to the dividend yield. In these equilibria the agent maintains a short position in the risky asset ($x^* < 0$) and again has negative wealth return. Equilibrium S_2 for the linear investment function in the left panel in Fig. 4.1 is of such kind. Finally, if $r^* \in (0, +\infty)$ the price return is positive, the agent position is characterized by a fixed fraction of wealth invested in the risky asset $x^* \in (0, 1)$ and his wealth return is positive. This is the case of equilibrium U_2 of the linear investment function and equilibria S_1 and U_1 of the nonlinear function.

After complete characterization of the location of equilibria, as the next natural step, we move to discuss the stability analysis of possible equilibria. The results are presented in the following

Proposition 4.3.2. *Fixed point* (x^*, r^*) *of the deterministic skeleton of system* (4.3.2) *is (locally) asymptotically stable if*

$$\frac{f'(r^*)}{l'(r^*)}\,\frac{1}{r^*} < 1\,, \qquad \frac{f'(r^*)}{l'(r^*)} < 1 \qquad \textit{and} \qquad \frac{f'(r^*)}{l'(r^*)}\,\frac{2+r^*}{r^*} > -1 \quad . \tag{4.3.4}$$

where $f'(r^*)$ *and* $l'(r^*) = \bar{e}/(\bar{e}+r^*)^2$ *stand for the first derivatives of the investment function* $f(r)$ *and of the EML* $l(r)$ *computed in equilibrium, respectively.*

The equilibrium is unstable if at least one of the inequalities in (4.3.4) *holds with the opposite (strict) sign. The stability is lost through* Neimark-Sacker*,* fold *or* flip *bifurcation if the first, the second or the third inequality in* (4.3.4)*, respectively, is violated.*

[2]Remember that the analysis is performed with respect to the rescaled variables as defined in (3.5.7). Negative return corresponds to the return less than r_f in terms of the unscaled variables.

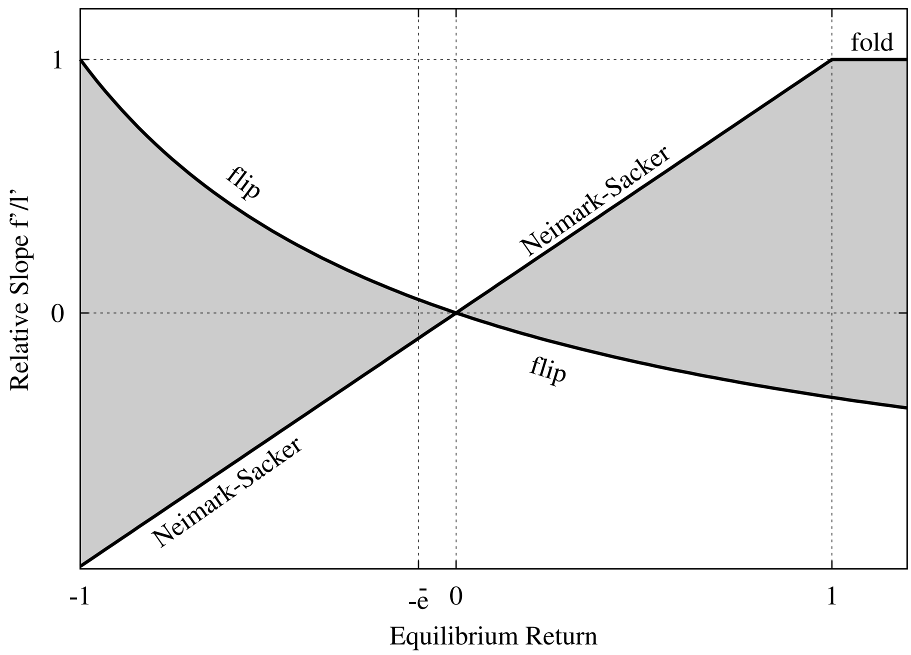

Figure 4.2: The stability region (gray) and the bifurcation types for the system (4.3.2) in parameter space with coordinates r^* and $f'(r^*)/l'(r^*)$.

Proof. See appendix D.2. □

This proposition defines the conditions for the asymptotic stability of equilibria and clarifies what type of bifurcation is exhibited when one of these conditions is violated. The two parameters relevant for the local asymptotical stability of equilibria are r^* and the relative slope of the investment function w.r.t. the EML, $f'(r^*)/l'(r^*)$. The stability region in terms of these two parameters is shown in Fig. 4.2 as gray area. Notice that the second inequality in (4.3.4) requires the slope of the investment function to be smaller than the slope of the EML. Therefore, equilibria U_1 and U_2 in Fig. 4.1 can be immediately recognized as unstable.

To understand better the local dynamics generated by the simplest possible version of our model, and, in particular, to study the effect of the slope of function f on the fixed point stability, let us observe the impact of system (4.3.2) on a sudden positive shock in the return. We will assume that investment function f is increasing in the fixed point, the case of the decreasing function being similar. We suppose that the system was in the fixed point (x^*, r^*), when the return suddenly increases on amount $\Delta r > 0$. At this moment period $t = 0$ starts. The further trajectories of the investment share and return together with the phase diagram are depicted in Fig. 4.3. The left panels show the dynamics for the stable situation with relatively flat investment function, whereas the opposite, unstable case with relatively steep investment function is shown in the right panels.

In both cases at period $t = 0$ the agent's expectations about future return are relatively high and equal to the previous return realization $r^* + \Delta r$. It leads to a higher (than in equilibrium) value of the agent investment share $x_0 \simeq x^* + f' \Delta r$, where f' is the slope of the investment function in the fixed point. Since the return is an increasing function of the present investment choice, the realized return r_0 is also greater than equilibrium level r^*. The further dynamics crucially depends on the amplification effect, based on the feedback mechanism linking investment share and market return. In other words, the long-run behavior depends

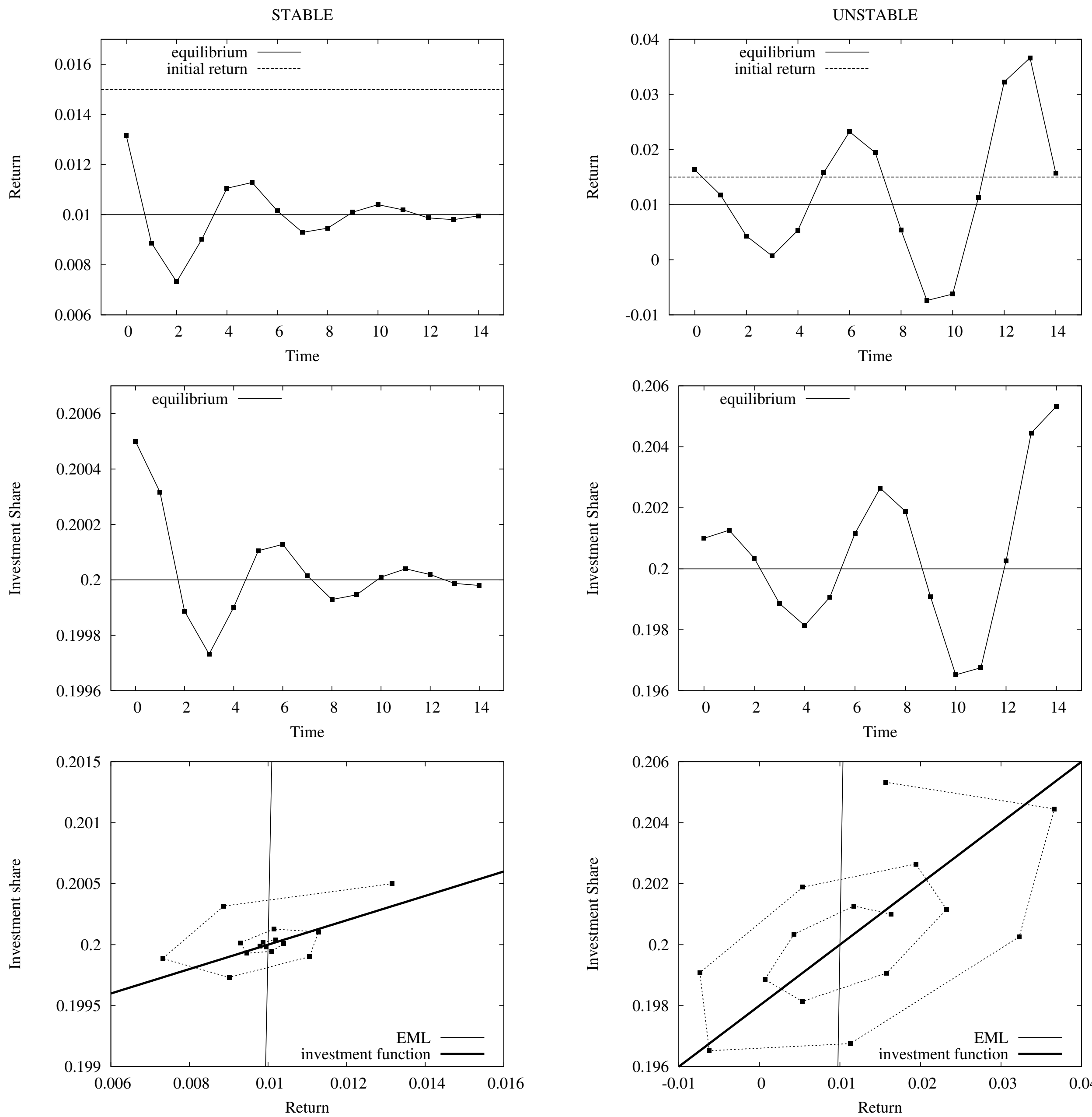

Figure 4.3: Trajectories for system (4.3.2) with linear investment function. Equilibrium is perturbed by the positive return shock. For relatively flat investment function with slope $f' = 0.1$ both return (**upper left panel**) and investment share (**middle left panel**) stabilize through the oscillations. With relatively steep investment function with slope $f' = 0.2$, the oscillations of return (**upper right panel**) and investment share (**middle right panel**) are reinforcing and the system does not stabilize. The phase plots with counter clock wise movement along the trajectories are shown in the **lower panels**. In both cases $x^* = 0.2, r^* = 0.01, \bar{e} = 0.04$ and shock $\Delta r = 0.005$.

on whether the initial shock in the return has been amplified by the investment behavior, i.e. whether $r_0 > r^* + \Delta r$, or not. It depends, of course, on the magnitude of increase in the investment share as a response on the change in expectations. Such magnitude, in turn, depends on the slope of function f.

In the left panels of Fig. 4.3 the investment function is flat and, consequently, r_0 is smaller than $r^* + \Delta r$, while in the right panels the situation is opposite. At time $t = 1$ the investment share decreases in the stable case as a response for the decrease in the return with respect to r_0. It leads to the stabilization of the system through oscillations. In the unstable case the investment share increases, $x_1 > x_0$ and consequent oscillations are destabilizing. Notice that even if in both cases $r_1 < r_0$, this, in principle, stabilizing decrease in the return is much higher in the case with flat investment function.

Fig. 4.2 also shows the types of bifurcation exhibited when one of the stability conditions is violated. For relatively small absolute values of the equilibrium return the system can loose stability through either flip or Neimark-Sacker bifurcation. In this case, if the system is perturbed away from equilibrium, the large absolute value of f' is responsible for an amplification of this perturbation as we show in the right panel of Fig. 4.3 for increasing investment function. For larger absolute values of the equilibrium return, if the investment function is steeper at the equilibrium than the EML, the equilibrium is lost through fold bifurcation, which implies a local exponential growth of the price returns.

4.3.2 Investment based on forecasted return

We now generalize the "naïve" forecast case and consider the situation in which agents use more sophisticated forecasting rule and base their investment on the EWMA estimate for the return computed according to the first equation in (4.2.2). Substituting (3.6.4) into the first recursive relation in (4.2.3) we reduce the market evolution to a two-dimensional stochastic system

$$\begin{cases} x_{t+1} = f(y_t) \\ y_{t+1} = \lambda y_t + (1-\lambda)\,\dfrac{f(y_t) - x_t + e_{t+1}\, x_t\, f(y_t)}{x_t\,(1 - f(y_t))} \end{cases} . \tag{4.3.5}$$

Even if the expression for the price return is not explicitly presented in this system, r_{t+1} is provided by the fraction in the right-hand side of the second equation.

As in the previous Section, we substitute the realizations of the yield process by its mean value $\bar{e}$ and consider the fixed points of the resulting *deterministic* skeleton. Their existence and location are characterized by the following statement which is completely analogous to Proposition 4.3.1

Proposition 4.3.3. *Let (x^*, y^*) be a fixed point of the deterministic skeleton of system* (4.3.5) *with price return r^*. Then it is:*

(i) The equilibrium return r^* and the equilibrium investment share x^* satisfy

$$l(r^*) = f(r^*)\,, \qquad x^* = f(r^*) \quad , \tag{4.3.6}$$

and the equilibrium value of predictor coincides with the equilibrium price return, $y^* = r^*$.

(ii) The equilibrium is feasible, i.e. the equilibrium prices are positive, if either $x^* < 1$ or $x^* \geq 1/(1-\bar{e})$.

(iii) The equilibrium growth rate of agent's wealth is equal to price return, $\rho^* = r^*$.

The dynamics of system (4.3.5) is qualitatively the same (at least locally, near the fixed points) to the dynamics in the case of naïve forecast. In particular, all equilibria belong to the Equilibrium Market Line and plot in Fig. 4.1 can be used for equilibria characterization without any modifications. We also found that at equilibrium the realized price return r^* coincides with the prediction of the EWMA estimator y^*. This is an important consistency result. In general, any meaningful economic dynamics should avoid equilibria with the systematic mistakes by the side of traders.

Finally, notice that as in the previous case, the set of equilibria is characterized both by exogenous dividend yield $\bar{e}$ and, as a consequence of our endogenous setting, by the agent's investment function f. On the other hand, equilibria are independent from the agent's forecast, i.e. from parameter λ. This parameter is important, however, in deciding the stability of equilibria, as we show in the following

Proposition 4.3.4. *Fixed point* (x^*, y^*) *of the deterministic skeleton of system* (4.3.5) *is (locally) asymptotically stable if*

$$\frac{f'(r^*)}{l'(r^*)}\,\frac{1}{r^*} < \frac{1}{1-\lambda}\,, \qquad \frac{f'(r^*)}{l'(r^*)} < 1 \qquad \textit{and} \qquad \frac{f'(r^*)}{l'(r^*)}\,\frac{2+r^*}{r^*} > -\,\frac{1+\lambda}{1-\lambda}\quad . \tag{4.3.7}$$

where $f'(r^*)$ *and* $l'(r^*) = \bar{e}/(\bar{e}+r^*)^2$ *stand for the first derivative of the investment function* $f(y)$ *and of the EML* $l(r)$ *computed in equilibrium, respectively.*

The equilibrium is unstable if at least one of the inequalities in (4.3.7) *holds with the opposite (strict) sign. The stability is lost through* Neimark-Sacker, fold *or* flip *bifurcation if the first, the second or the third inequality in* (4.3.7), *respectively, is violated.*

Proof. See appendix D.3. □

Stability conditions (4.3.7) are similar to the conditions obtained for naïve forecast and reduce to them when $\lambda = 0$. Two different regions defined by (4.3.7) with two different values of parameter λ are shown in Fig. 4.4.

Since the locations of equilibria do not depend on λ, it immediately follows from Proposition 4.3.4 that an increase in the value of λ, that is, loosely speaking, in the agent's memory length, brings stability into the system. This effect can be easily understood comparing the EWMA forecast (4.2.2) with the naïve forecast: any shock in the return leads to smaller changes in the expected future return when the former is used instead of the latter. As a result, any fixed point (except, possibly, the ones with $y^* = 0$) becomes stable when λ takes a sufficiently large value. For example, both equilibria S_1 and S_2 in Fig. 4.1 are stable for sufficiently large λ.

4.3.3 Investment based on forecasted return and variance

Let us now move to the case of a single agent whose investment function depends on both estimators defined in Assumption 2′. One can see this case as an extension of the previous analysis to agent who decides his investment taking into account not only expected profit, but also the endogenous component of risk involved in his portfolio decisions. Indeed, notice that (3.6.5) explicitly identifies two different risk components in the profit coming from investment in the risky asset: stochastic yield process $\{e_t\}$ and possibly volatile return dynamics $\{r_t\}$. The former is assumed exogenous (Assumption 1) and perfectly known to agents (Assumption 2′). The latter is, on the contrary, endogenous and not perfectly known. EWMA estimator z_t

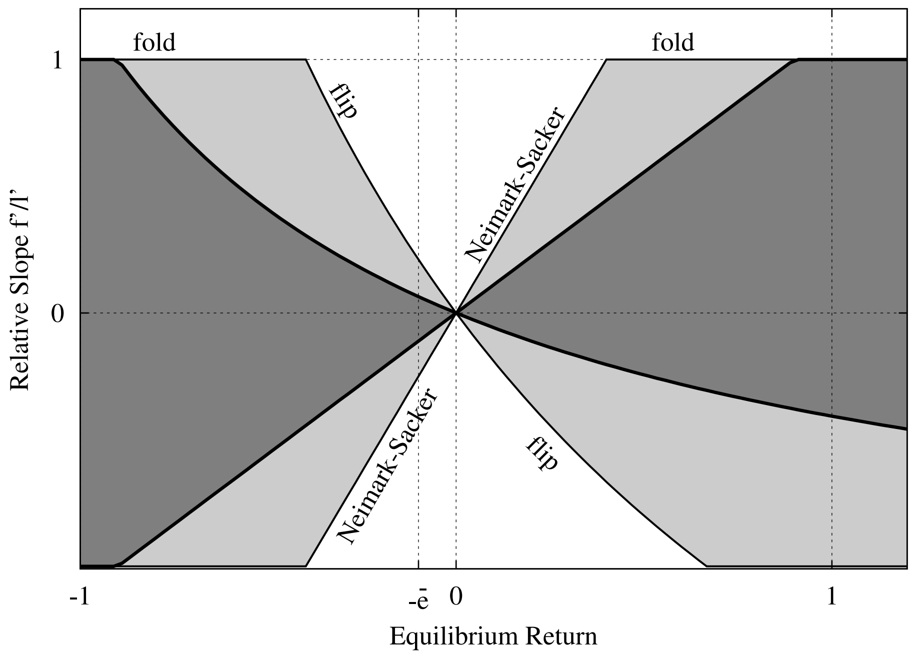

Figure 4.4: Stability region for two different values of λ in coordinates r^* and $f'(r^*)/l'(r^*)$. For $\lambda = 0.1$, the fixed point is locally stable if $(r^*, f'/l')$ belongs to the dark-gray region. When λ increases to the value 0.6, the stability region expands and becomes the union of the dark-gray and light-gray areas.

for the variance of future return introduced in (4.2.2) can be thought of as a measure of this second component of risk.

Combining (4.2.1), (4.2.3) and (3.6.4), one gets the following three-dimensional system:

$$\begin{cases} x_{t+1} = f(y_t, z_t) \\ y_{t+1} = \lambda y_t + (1-\lambda)\,\dfrac{f(y_t, z_t) - x_t + e_{t+1}\, x_t\, f(y_t, z_t)}{x_t\,(1 - f(y_t, z_t))} \\ z_{t+1} = \lambda z_t + \lambda(1-\lambda)\left[\dfrac{f(y_t, z_t) - x_t + e_{t+1}\, x_t\, f(y_t, z_t)}{x_t\,(1 - f(y_t, z_t))} - y_t\right]^2 \end{cases} . \tag{4.3.8}$$

It is easy to see that the EML can still be used for the characterization of equilibria. Namely, the following applies

Proposition 4.3.5. *Let (x^*, y^*, z^*) be a fixed point of the deterministic skeleton of system* (4.3.8) *and let us denote with r^* the price return in this point. Then it is:*

(i) The equilibrium return r^* and the equilibrium investment share x^* satisfy

$$l(r^*) = f(r^*, 0)\ , \qquad x^* = f(r^*, 0) \quad . \tag{4.3.9}$$

The equilibrium value of the return estimator coincides with the equilibrium price return, $y^* = r^*$, while the equilibrium value of the variance estimator is zero, $z^* = 0$.

(ii) The equilibrium is feasible, if either $x^* < 1$ or $x^* \geq 1/(1-\bar{e})$.

(iii) The equilibrium growth rate of the agent's wealth is equal to the equilibrium price return, $\rho^* = r^*$.

All the differences between this statement and Proposition 4.3.3 are in the first item. The consistency result for the equilibrium return estimator $y^* = r^*$ is confirmed and extended to the variance estimator z^*, whose value at equilibrium becomes zero, as expected for a constant return, i.e. geometrically increasing price dynamics. The conditions characterizing the equilibrium investment share and return have slightly changed, because investment function f now depends on two variables. However, if one considers the restriction of f on the set $z = 0$, it is clear that the former characterization of equilibria as the intersections with the EML is still valid.

The following result provides the conditions for the stability of the equilibria.

Proposition 4.3.6. *The fixed point (x^*, y^*, z^*) of the deterministic skeleton of system* (4.3.8) *is (locally) asymptotically stable if*

$$\frac{f'_y(r^*,0)}{l'(r^*)}\frac{1}{r^*} < \frac{1}{1-\lambda}, \qquad \frac{f'_y(r^*,0)}{l'(r^*)} < 1 \qquad \textit{and} \qquad \frac{f'_y(r^*,0)}{l'(r^*)}\frac{2+r^*}{r^*} > -\frac{1+\lambda}{1-\lambda} \quad , \tag{4.3.10}$$

where f'_y is, with usual notation, the partial derivative of the investment function with respect to the first variable y and l' is the first derivative of the EML.

The equilibrium is unstable if at least one of the inequalities in (4.3.10) *holds with the opposite (strict) sign. The system exhibits a* Neimark-Sacker*,* fold *or* flip *bifurcation if the first, the second or the third inequality in* (4.3.7)*, respectively, becomes an equality.*

Proof. See appendix D.4. □

Since the partial derivative f'_y is exactly the derivative of the restriction of the investment function f on the set $z = 0$, it is clear that by taking this restriction the last Proposition reduced to Proposition 4.3.4.

The most interesting conclusion of the analysis of this extension with respect to the model discussed in Section 4.3.2 is that the introduction of a measure of endogenous risk in the agent's investment function, albeit changing, in general, the global behavior of the agent and of the resulting system, does not have any impact on the local dynamics in the neighborhood of an equilibrium[3]. This is not to say that investor's attitude towards risk is irrelevant for the stability of the system. Notice that risk aversion is a property of the specific functional form of the investment function, something that cannot be investigated here. The point of the last two Propositions is that for any investment function $f(y, z)$ there exists an "equivalent" function depending on the sole estimation y, namely $f(y, 0)$, that provides all the information concerning the allowed equilibria and their stability. Consequently, the geometrical illustrations of the location of equilibria presented in Fig. 4.1 and stability conditions in Fig. 4.4 remain valid when one considers the restricted function $f(y, 0)$. As an example in Section 4.5.2 will demonstrate, if the degree of risk aversion is expressed as the value of some parameter in a given functional specification of f, it will be maintained also in its restriction to the $z = 0$ plane.

In this Section we thoroughly analyzed the single agent case. We found that our model generates speculative equilibria which can be geometrically characterized through the Equilibrium Market Line plot like in Fig. 4.1. We found that equilibria with $r^* + \bar{e} = 0$ are impossible.

[3]This result is similar to what found in Gaunersdorfer (2000), in Bottazzi (2002) and in the model of Chapter 2 for CARA-types of investors.

We also derived stability conditions for equilibria and found that with sufficiently high value of λ any equilibrium in which investment strategy intersects the EML *from below* will become stable.

4.4 Economy with Heterogeneous Agents

In this Section we consider the case in which many heterogeneous agents, with different investment functions, operate in the market. According to Assumption $2'$ all these functions depend on the EWMA estimators of future price dynamics, but we allow the parameters of these estimators to differ among agents. This Section is divided into three parts. First, we write down the system of the difference equations and introduce necessary notation. Second, we compute all equilibria of such system and show as the Equilibrium Market Line introduced in Definition 4.3.1 can be used for their geometrical characterization. Third, we provide the stability conditions for all equilibria.

4.4.1 Dynamical system

The main difference of the heterogeneous setting with respect to the single agent case concerns the role of the wealth dynamics. Indeed, the evolution of wealth shares is no longer decoupled from the dynamics of price and, consequently, both (3.6.4) and the entire set of equations in (3.6.6) become relevant.

Under the conditions in Assumption $2'$ the evolution of the economy can be described in terms of the variables $x_{t,n}$, $y_{t,n}$ and $z_{t,n}$ for $n \in \{1, \dots, N\}$ and of $\varphi_{t,n}$ for $n \in \{1, \dots, N-1\}$ as in the following

Lemma 4.4.1. *The dynamics defined by* (3.6.4) *and* (3.6.6) *with investment choices* (4.2.1) *can be described by means of the following system of* $4N-1$ *first-order equations*

$$
\begin{aligned}
\mathcal{X} : &\left[\begin{array}{lcl}
x_{t+1,1} & = & f_1\big(y_{t,1}, z_{t,1}\big) \\
\quad\vdots & \vdots & \vdots \\
x_{t+1,N} & = & f_N\big(y_{t,N}, z_{t,N}\big)
\end{array}\right. \\
\mathcal{Y} : &\left[\begin{array}{lcl}
y_{t+1,1} & = & \lambda_1\, y_{t,1} + (1-\lambda_1)\, r_{t+1} \\
\quad\vdots & \vdots & \vdots \\
y_{t+1,N} & = & \lambda_N\, y_{t,N} + (1-\lambda_N)\, r_{t+1}
\end{array}\right. \\
\mathcal{Z} : &\left[\begin{array}{lcl}
z_{t+1,1} & = & \lambda_1\, z_{t,1} + \lambda_1\,(1-\lambda_1)\,\big(r_{t+1} - y_{t,1}\big)^2 \\
\quad\vdots & \vdots & \vdots \\
z_{t+1,N} & = & \lambda_N\, z_{t,N} + \lambda_N\,(1-\lambda_N)\,\big(r_{t+1} - z_{t,N}\big)^2
\end{array}\right. \\
\mathcal{W} : &\left[\begin{array}{lcl}
\varphi_{t+1,1} & = & \varPhi_1\big(x_{t,1}, \dots, x_{t,N}; \varphi_{t,1}, \dots, \varphi_{t,N-1}; e_{t+1}; r_{t+1}\big) \\
\quad\vdots & \vdots & \vdots \\
\varphi_{t+1,N-1} & = & \varPhi_{N-1}\big(x_{t,1}, \dots, x_{t,N}; \varphi_{t,1}, \dots, \varphi_{t,N-1}; e_{t+1}; r_{t+1}\big)
\end{array}\right.
\end{aligned}
\tag{4.4.1}
$$

where

$$
\varPhi_n\Big(x_1, x_2, \dots, x_N; \varphi_1, \varphi_2, \dots, \varphi_{N-1}; e; r\Big) = \varphi_n\,\frac{1 + (r+e)\,x_n}{1 + (r+e)\sum_{m=1}^{N} \varphi_m\, x_m} \quad , \tag{4.4.2}
$$

for $n \in \{1,\dots,N-1\}$, and where the price return r_{t+1} reads

$$r_{t+1} = \frac{\sum_{n=1}^{N} \varphi_{t,n} \Big(f_n(y_{t,n}, z_{t,n}) \big(1 + e_{t+1}\, x_{t,n}\big) - x_{t,n} \Big)}{\sum_{n=1}^{N} \varphi_{t,n}\, x_{t,n} \big(1 - f_n(y_{t,n}, z_{t,n})\big)} \quad . \tag{4.4.3}$$

with

$$\varphi_{t,N} = 1 - \sum_{n=1}^{N-1} \varphi_{t,n} \quad .$$

The equations are ordered to obtain four separated blocks: $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{W}$. The N equations in block $\mathcal{X}$ give the investment choices of the N agents, accordingly to (4.2.1). Blocks $\mathcal{Y}$ and $\mathcal{Z}$ contain the N recursive relations (4.2.3) describing the evolution of the EWMA estimates of return and its variance, respectively, for the different agents. The evolution of the wealth shares is described by the equations in block $\mathcal{W}$; notice that the number of independent wealth shares in the system is $N-1$. The evolution of price return is provided by (4.4.3) in accordance with (3.6.4).

The rest of this Section is devoted to the analysis of the *deterministic skeleton* of (4.4.1) obtained replacing the yield process $\{e_t\}$ by its mean value $\bar{e}$.

4.4.2 Determination of equilibria

Let us denote an equilibrium of system (4.4.1) as

$$\boldsymbol{x}^* = \left(x_1^*, \dots, x_N^*; y_1^*, \dots, y_N^*; z_1^*, \dots, z_N^*; \varphi_1^*, \dots, \varphi_{N-1}^*\right) \quad ,$$

and let r^* be the associated equilibrium return. We introduce the following

Definition 4.4.1. Agent n is said to *"survive"* in $\boldsymbol{x}^*$ if his equilibrium wealth share is strictly positive, $\varphi_n^* > 0$. Agent n is said to *"dominate"* agent n' in $\boldsymbol{x}^*$ if $\varphi_{n'}^*/\varphi_n^* = 0$. An agent n who dominates, at equilibrium, any other agent $n' \neq n$ is said to *"dominate"* the economy.

Similar conceptions of survival and dominance have been introduced inside different settings in the models of DeLong, Shleifer, Summers, and Waldmann (1991) and Blume and Easley (1992). These models are built in the probabilistic framework, while we adopt here the deterministic version of the concepts of survival and dominance.

The characterization of the fixed points of system (4.4.1) is, in many respects, similar to the single agent case discussed in Section 4.3. There is, however, one important difference. Remember that in the market with single agent, the equilibrium return could not be equal to $-\bar{e}$. The EML had a vertical asymptote in this point. It turns out that for many agent case there are equilibria with $r^* = -\bar{e}$. We call them **"no-arbitrage" equilibria**, since two assets have there the same expected return. Remember, that zero rescaled price return corresponds to the risk-free return of the risky asset. We will refer to those equilibria where $r^* \neq -\bar{e}$ as **"EML" equilibria**, since they can be identified by means of the Equilibrium Market Line.

Consequently, we perform our analysis in this Section in two steps. In the first step we look for the EML equilibria. They can be characterized in the similar way with equilibria for the single agent case. Then, we discuss the no-arbitrage equilibria. In particular, we will notice that for their existence some agents have to go short in the risky asset, i.e. have negative x's.

EML equilibria

All possible equilibria of system (4.4.1) with the restriction $r^* \neq -\bar{e}$ can be characterized by the following statement.

Proposition 4.4.1. *Let* $\boldsymbol{x}^*$ *be a fixed point of the deterministic skeleton of system* (4.4.1)*, such that* $r^* \neq -\bar{e}$*. Then it is*

$$y_n^* = r^* \,, \qquad z_n^* = 0 \qquad \forall n \in \{1, \dots, N\} \quad , \tag{4.4.4}$$

and the following two mutually exclusive cases are possible:

(i) ***Single agent survival.*** *In* $\boldsymbol{x}^*$ *only one agent survives and, therefore, dominates the economy. Without loss of generality we can assume this agent to be agent* 1 *so that for the equilibrium wealth shares one has*

$$\varphi_n^* = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \quad . \tag{4.4.5}$$

The equilibrium return r^* *is determined as the solution of*

$$l(r^*) = f_1(r^*, 0) \quad , \tag{4.4.6}$$

while the equilibrium investment shares are defined according to

$$x_n^* = f_n(r^*, 0) \qquad \forall n \in \{1, \dots, N\} \quad . \tag{4.4.7}$$

The survivor's wealth growth rate equal to the equilibrium price return, $\rho_1^* = r^*$*.*

(ii) ***Many agents survival.*** *In* $\boldsymbol{x}^*$ *more than one agent survives. Without loss of generality we can assume that the survivors are the first* k *agents (with* $k > 1$*) so that the equilibrium wealth shares satisfy*

$$\begin{cases} \varphi_n^* \in (0,1) & \text{if } n \leq k \,, \\ \varphi_n^* = 0 & \text{if } n > k \end{cases} \quad , \qquad \sum_{n=1}^{k} \varphi_n^* = 1 \quad . \tag{4.4.8}$$

The equilibrium return r^* *satisfies the following* k *equations*

$$l(r^*) = f_n(r^*, 0) \qquad \forall n \in \{1, \dots, k\} \quad . \tag{4.4.9}$$

The equilibrium investment shares are defined according to

$$x_n^* = f_n(r^*, 0) \qquad \forall n \in \{1, \dots, N\} \quad , \tag{4.4.10}$$

so that the first k *agents possess, at equilibrium, the same investment share* $x_{1\diamond k}^* = l(r^*)$*.*

The wealth growth rates of the survivors are, at equilibrium, equal to the equilibrium price return, i.e. $\rho_n^* = r^*$ *for* $n \leq k$*.*

Proof. See Appendix D.5. □

Strictly speaking, item *(i)* of the previous Proposition can be seen as a particular case of item *(ii)*. Nevertheless, the nature of the two situations is deeply different. In the first case, when a single agent survives, Proposition 4.4.1 defines a precise value for each component of the equilibrium $\boldsymbol{x}^*$, so that a single point is uniquely determined. In the second case, on the contrary, there is a residual degree of freedom in the definition of the equilibrium: while investment shares x^*'s and estimates y^*'s and z^*'s are uniquely defined, the only requirement on the equilibrium wealth shares of the surviving agents is the fulfillment of the second equality in (4.4.8). Consequently, one immediately has the following

Corollary 4.4.1. *Consider the deterministic skeleton of system* (4.4.1). *If it possesses an equilibrium* $\boldsymbol{x}^*$ *with* k *survivors it possesses a* $k-1$*-simplex of* k*-survivors equilibria constituted by all the points obtained from* $\boldsymbol{x}^*$ *through a change in the relative wealths of the survivors.*

Thus, Proposition 4.4.1*(ii)* does not define a single equilibrium point, but an infinite set of equilibria. If the survivors are the first k agents as in (4.4.8), this set can be written as

$$\left\{ \Big(x_1^*, \ldots, x_N^*; \underbrace{r^*, \ldots, r^*}_{N}; \underbrace{0, \ldots, 0}_{N}; a_1, \ldots, a_k, \underbrace{0, \ldots, 0}_{N-1-k} \Big) \;\Bigg|\; \sum_{j=1}^{k} a_j = 1 \;, \quad a_j > 0 \right\}$$

The particular fixed point eventually chosen by the system will depend on the initial conditions. We will see below that the partially indeterminate nature of the many survivors equilibria will have a major effect also on their asymptotic stability.

The differences among the two cases of Proposition 4.4.1 does not only regard the geometric nature of the *locus* of equilibria. Indeed, while in the first case no requirements are imposed on the behavior of the investment functions of the different agents, in the second type of solution all the investment shares $x_1^*, \ldots, x_k^*$ must *at the same time* be equal to a single value $x_{1\diamond k}^*$. Thus, the equilibrium with $k > 1$ survivors exists only in the particular case in which k investment functions $f_1, \ldots, f_k$ satisfy this restriction. This implies that an economy composed by N agents having generic, so to speak "randomly defined", investment functions, has probability zero of displaying any equilibrium with multiple survivors. In other terms, the many survivors equilibria are non-generic.

Both types of multi-agent equilibria derived in Proposition 4.4.1 are strictly related to the "special" single-agent equilibria. As in the single agent case, the growth rate of the total wealth is equal to the equilibrium price return and is determined by the growth rate of those agents who survive in the equilibrium. At the same time, due to (4.4.4), the requirement of estimators consistency in the equilibrium is satisfied for any agent. Finally, the determination of the equilibrium return level r^* for the multi-agent case in (4.4.6) or (4.4.9) is identical to the case where the agent, or one of the agents, who would survive in the multi-agent equilibrium, is present alone in the market. An useful consequence of this fact is that the geometrical interpretation of market equilibria presented in Section 4.3 can be extended to illustrate how equilibria with many agents are determined.

As an example consider again Fig. 4.1 and suppose that the two investment functions shown there[4] belong to two agents who are simultaneously operating in the market. According to Proposition 4.4.1 all possible equilibria are the intersections of these functions with the Equilibrium Market Line (c.f. (4.4.6) and (4.4.9)). In this example there are four possible equilibria. In two of them (S_1 and U_1) the first agent, with non-linear investment function,

[4]With some abuse of language, we use the same term "investment function" for the corresponding restrictions of the actual investment functions. See the discussion in Section 4.3.3.

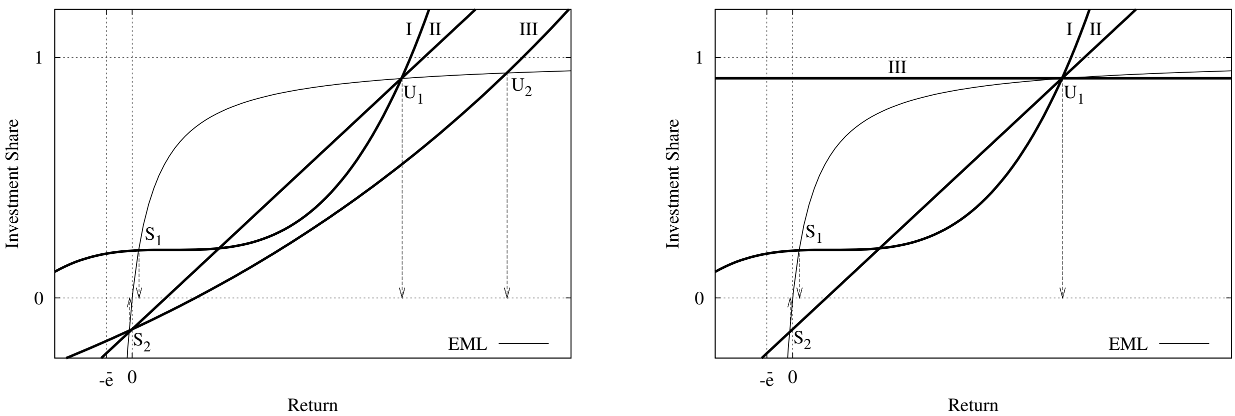

Figure 4.5: Non-generic situations with 3 agents operating on the market. **Left panel**: In equilibria S_2 and U_1 two agents survive. **Right panel**: In equilibrium U_1 all three agents survive.

survives such that $\varphi_1^* = 1$ (and obviously $\varphi_2^* = 0$). In the other two equilibria (S_2 and U_2) the second agent, with linear investment function, survives so that, in these points, $\varphi_1^* = 0$. In each equilibrium, the intersection of the investment function of the surviving agent with the Equilibrium Market Line gives both the equilibrium return r^* and the equilibrium investment share of the survivor. The equilibrium investment share of the non-surviving agent can be found, in accordance with (4.4.10), as the intersection of his own investment function with the vertical line passing through the equilibrium return. Since the two investment functions shown in Fig. 4.1 do not possess common intersections with the EML, equilibria with more than one survivors are impossible in this case.

Two examples of investment functions which allow for multiple survivors equilibria are reported in Fig. 4.5. The abscissa of common intersections of different investment functions with the EML defines the return in multiple survivors equilibria. In each equilibrium, all the survivors invest the same share of wealth $x^*_{1\diamond k}$, determined as the ordinate of the intersection. The wealth shares of survivors should satisfy to (4.4.8). The non surviving agents (like agent with investment function I in equilibrium S_2 or agent with function III in equilibrium U_1 in the left panel of Fig. 4.5) have zero wealth shares. Their investment shares are the intersections of their investment functions with the vertical line passing through the equilibrium return.

No-arbitrage equilibria

To complete the description of possible market equilibria, let us move to the case which is completely new with respect to the single agent scenario. From system (4.4.1) and relation (4.4.3) it is straight-forward to see that the following applies

Proposition 4.4.2. *Let $\boldsymbol{x}^*$ be a fixed point of the deterministic skeleton of system* (4.4.1) *with $r^* = -\bar{e}$. Then, in addition to the consistency requirement* (4.4.4)*, it is*

$$\sum_{n=1}^{N} x_n^* \, \varphi_n^* = 0 \quad , \tag{4.4.11}$$

where equilibrium investment shares are defined as

$$x_n^* = f_n(-\bar{e}, 0) \qquad \forall n \in \{1, \dots, N\} \quad . \tag{4.4.12}$$

The wealth growth rates of all agents are zero in such equilibria.

This proposition shows that when the agents' investment shares are "balanced" by their wealth shares, one can get equilibria where capital gain and dividend yield offset each other. In these equilibria many agents coexist and, therefore, survive in the long-run. As opposite to the situation described in Proposition 4.4.1*(ii)*, this many survivors equilibrium can be considered "generic". The only requirement for the existence of such equilibria is that there are two agents in the market, one with positive and one with negative position in the risky asset. That is the reason why we did not have such no-arbitrage equilibria in the single agent case. When the number of agents in the economy become large, there emerge more possibilities for the existence of the no-arbitrage equilibria. We formalize it in the following

Corollary 4.4.2. *Consider the deterministic skeleton of system* (4.4.1). *This skeleton possesses no-arbitrage equilibrium* $\boldsymbol{x^*}$ if and only if *there are two agents i and j whose investment functions satisfy to the following condition*

$$f_i(-\bar{e},0)\ f_j(-\bar{e},0) < 0 \quad . \tag{4.4.13}$$

If this condition is satisfied and $N = 2$, then such equilibrium is unique. If this condition is satisfied and $N > 2$, then the skeleton possesses the following $N-2$-dimensional manifold of many survivors equilibria

$$\left\{ \Big(x_1^*, \ldots, x_N^*; \underbrace{-\bar{e}, \ldots, -\bar{e}}_{N}; \underbrace{0, \ldots, 0}_{N}; a_1, \ldots, a_{N-1-k} \Big) \quad \Bigg| \quad \sum_{j=1}^{N} a_j = 1 \ , \sum_{j=1}^{N} a_j x_j^* = 0 \ , a_j \geq 0 \right\} \quad .$$

Proof. See appendix D.6. □

The equilibrium manifold introduced in this Corollary is constituted by all the points obtained from $\boldsymbol{x^*}$ through a change in the wealth shares of the agents in the direction orthogonal to the vector of the agents' investment shares.

We illustrate new equilibria in Fig. 4.6. The left panel represents our old example of Fig 4.1. We already found four EML equilibria in such market with one survivor in each of them. In addition, there is another equilibrium with $r^* = -\bar{e}$ which geometrically can be represented by two points A_1 and A_2 showing the corresponding investment shares of the agents. In the right panel we show the situation with three agents operating in the market. Only one of these agents survives in equilibria S_1, S_2, S_3 and U_2. There exist also two equilibria with two survivors. In one of them the agent with positive constant strategy invests in A_1 and the agent with negative constant strategy invests in A_2. Another equilibrium is the one, where the agent with increasing strategy invests in A_3 and the agent with negative constant strategy invests in A_2. Notice that there is no equilibrium determined by couple of points A_1 and A_3, since in both of them the agents' investment shares are positive. Finally, the system possesses a one-dimensional manifold of no-arbitrage equilibria where all three agents survive.

4.4.3 Stability conditions of equilibria

Among the different equilibria characterized in the previous Section, which are the ones eventually selected by the market? This Section answers this question presenting the results of the local stability analysis for all possible equilibria. There are three main propositions here,

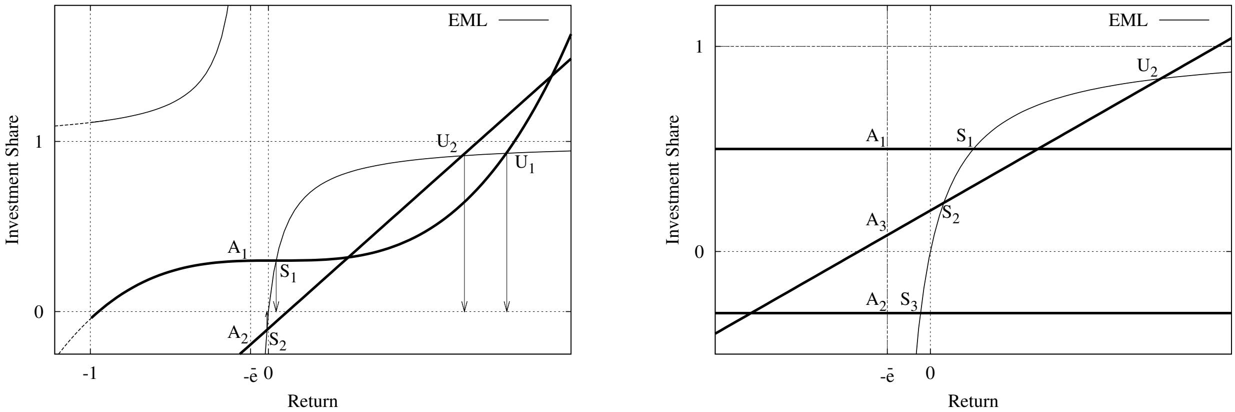

Figure 4.6: Illustration of no-arbitrage equilibria with $r^* = -\bar{e}$. In addition to the EML equilibria S_1, S_2, U_1 and U_2, market in the **left panel** possesses an equilibrium, represented by points A_1 and A_2 with two survivors. Market in the **right panel** has four EML equilibria with single survivor, two no-arbitrage equilibria (represented by couples A_1 with A_2 and A_3 with A_2) with two survivors, and manifold of no-arbitrage equilibria with three survivors.

one for each type of equilibria which we identified in Section 4.4.2. The first Proposition provides the stability region in the parameter space for the EML equilibria in generic case of one single survivor. The non-generic case of many survivors, when equilibrium return $r^* \neq -\bar{e}$, is addressed in the second Proposition. Finally, in the third Proposition we will derive stability conditions for the case with many survivors in no-arbitrage equilibrium with $r^* = -\bar{e}$. In particular, we show the destabilizing effect of the existence of an entire hyperplane of equilibria in all cases when there are many survivors.

The derivation of all Propositions of this Section requires quite cumbersome algebraic manipulations and we refer the reader to Appendix D.7 for the intermediate Lemmas and the final proofs. In this Section we provide only short comments for each Proposition. More extended comments and applications of these results are postponed to the next Section.

Equilibrium with single survivor

For the generic case of a single survivor equilibrium we have the following

Proposition 4.4.3. *Let $\boldsymbol{x^*}$ be a fixed point of the deterministic skeleton of system* (4.4.1) *associated with a single survivor equilibrium. Without loss of generality we can assume* (4.4.5), *i.e. that the first agent is the survivor. Then $\boldsymbol{x^*}$ is (locally) asymptotically stable if*

$$\frac{f'_{1,y}(r^*,0)}{l'(r^*)}\,\frac{1}{r^*} < \frac{1}{1-\lambda_1} \quad , \quad \frac{f'_{1,y}(r^*,0)}{l'(r^*)} < 1 \quad , \quad \frac{f'_{1,y}(r^*,0)}{l'(r^*)}\,\frac{2+r^*}{r^*} > -\,\frac{1+\lambda_1}{1-\lambda_1} \tag{4.4.14}$$

where $f'_{1,y}$ denotes the partial derivative of investment function f_1 with respect to the expected return y and if

$$-2 - r^* < x^*_n\left(r^* + \bar{e}\right) < r^*\,, \quad \forall n > 1 \quad . \tag{4.4.15}$$

The equilibrium $\boldsymbol{x^*}$ is unstable if at least one of the inequalities in (4.4.14) *or in* (4.4.15) *holds with the opposite (strict) sign.*

The system exhibits a Neimark-Sacker *bifurcation if the first inequality in* (4.4.14) *becomes an equality, a* fold *bifurcation if the second inequality in* (4.4.14) *or one of the* $N-1$ *right-hand inequalities in* (4.4.15) *becomes an equality and a* flip *bifurcation if the third inequality in* (4.4.14) *or one of the* $N-1$ *left-hand inequalities in* (4.4.15) *becomes an equality.*

The stability conditions for a generic fixed point contain two requirements. On the one hand, the stable equilibrium should be "self-consistent", i.e. it should remain stable even if any non-surviving agent is removed from the economy. Indeed, the three inequalities in (4.4.14) coincide with the corresponding conditions in (4.3.10) as if the survivor would operate alone in the market. This is, however, not enough. A further requirement comes from the two inequalities in (4.4.15). In particular, notice that in those equilibria where $r^* > -\bar{e}$, i.e. where the overall wealth of the economy grows, the surviving agent must be the most aggressive one, i.e. invest the highest wealth share in the risky asset among all agents. On the other hand, in those equilibria where $r^* < -\bar{e}$, i.e. in which the overall wealth of the economy shrinks, the survivor has to be the least aggressive.

For the single survivor equilibrium the role of the "memory" parameter λ is similar to what found for the single agent case in Proposition 4.3.6. The equilibrium stability domain increases with the value of the survivor's λ. In the case of heterogeneous agents, however, the scope of this statement is restricted by an important *caveats*: if the additional condition in (4.4.15) is not satisfied, the equilibrium remains unstable irrespectively of the value of λ.

Let us revert once again to the EML plot to obtain a geometrical illustration of the many agent stability conditions. In Fig. 4.7 we re-draw two investment functions from Fig. 4.1. The region where condition (4.4.15) is satisfied is reported in gray. Suppose that the market is populated by two agents whose investment decisions are respectively described by these two functions. Notice that they are the same functions appearing in Fig. 4.1 and that were discussed in Section 4.3. Proposition 4.4.1 allowed us to identify four possible equilibria: S_1, S_2, U_1 and U_2. First, notice that the dynamics of the two-agent system cannot be attracted by U_1 or U_2. Since these equilibria were unstable in the respective single agent cases, they cannot be stable when both agents are present in the market. Assume that S_1 and S_2 would be stable equilibria if the first and the second function, respectively, were present alone in the market. Then, from Proposition 4.4.3, it follows that S_1 is the only stable equilibrium of the system with two agents. Notice, indeed, that for a value of r equal to the abscissa of S_1, i.e. equal to the equilibrium return, the linear investment function of the non-surviving agent passes below the investment function of the surviving agent and belongs to the gray area. On the contrary, in the abscissa of S_2, the investment function of the non-surviving agent is higher and does not belong to the gray area. Consequently, the latter equilibrium is unstable.

Let us move now to the case of many survivors. They can survive either in "no arbitrage" equilibrium where $r^* = -\bar{e}$, or in the EML equilibrium with $r^* \neq -\bar{e}$. We start with the latter, non-generic case, which we illustrated in Fig. 4.5.

EML equilibria with many survivors

In the equilibria identified in Proposition 4.4.1*(ii)*, the following applies

Proposition 4.4.4. *Let* $\boldsymbol{x}^*$ *be a fixed point of the deterministic skeleton of system* (4.4.1) *belonging to a* $k-1$*-dimensional manifold of* k*-survivors equilibria defined by* (4.4.8), (4.4.9) *and* (4.4.10). *Without loss of generality we can assume* (4.4.8)*, so that first* k *agents survive.*

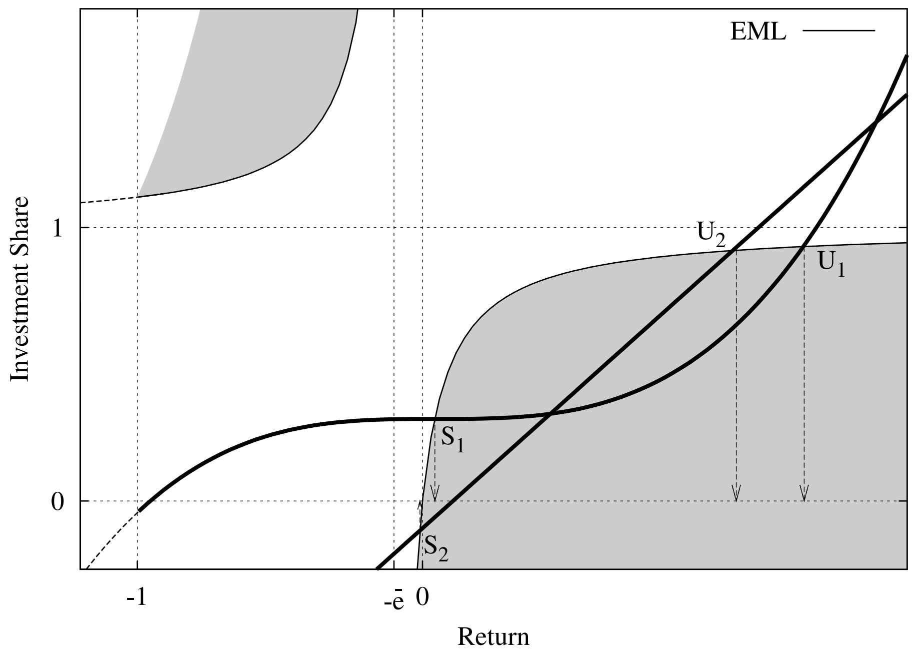

Figure 4.7: Equilibria and their stability conditions for the many-agents market. Any intersection of the investment function with the EML provides the equilibrium return (abscissa) and the equilibrium investment share of survivor (ordinate), while the ordinate of another investment function in this point gives the equilibrium investment share of non-surviving agent. The investment share of non-survivor should belong to the gray area in order the necessary condition (4.4.15) for the stability was satisfied.

The fixed point $\boldsymbol{x}^*$ *is never hyperbolic and, consequently, never (locally) asymptotically stable. Its non-hyperbolic submanifold is the* $k-1$*-simplex defined in Corollary 4.4.1.*

The equilibrium $\boldsymbol{x}^*$ *is (locally) stable if*

(i) all the roots of the following polynomial are inside the unit circle

$$\mu \prod_{j=1}^{k} (\lambda_j - \mu) + \frac{(1+r^*)\mu - 1}{x^*_{1\diamond k}(1 - x^*_{1\diamond k})} \sum_{j=1}^{k} \left(\varphi^*_j \, f'_{j,y}(1-\lambda_j) \prod_{i=1, i\neq j}^{k} (\lambda_i - \mu) \right) \quad , \tag{4.4.16}$$

where $f'_{j,y}$ *denotes the value of the partial derivative of investment function* f_j *with respect to the expected return* y *in the point* $(r^*, 0)$*;*

(ii) the equilibrium investment shares of the non-surviving agents satisfy

$$-2 - r^* < x^*_n (r^* + \bar{e}) < r^* \,, \quad k < n \leq N \,. \tag{4.4.17}$$

The equilibrium $\boldsymbol{x}^*$ *is unstable if at least one of the roots of polynomial in* (4.4.16) *is outside the unit circle or if at least one of the inequalities in* (4.4.17) *holds with the opposite (strict) sign.*

The non hyperbolic nature of the many survivor equilibria is a direct consequence of their non-unique specification in Proposition 4.4.1*(ii)*. The motion of the system along the $k-1$ dimensional subspace consisting of the continuum of equilibria leaves the aggregate properties of the system invariant, however.

The stability conditions for the many survivors equilibria generalize the stability conditions derived in Proposition 4.4.3 for the single survivor case. Indeed, it is immediate to see that the condition in Proposition 4.4.4*(i)* reduces to inequalities (4.4.14) when $k = 1$. The constraint on the investment shares (4.4.17) are identical to (4.4.15). Thus, also in the non-generic equilibria with $r^* > -\bar{e}$ survivors should be most aggressive agents, while in those equilibria where $r^* < -\bar{e}$ survivors should be the least aggressive.

From the stability condition in item *(i)* it follows that not all equilibria in the $k-1$ dimensional manifold are equivalent from the point of view of stability. Weighted structure of the second term of polynomial in (4.4.16) implies that stability of the equilibrium with many survivors may be lost when the wealth shares of these survivors are changing.

The last Proposition does not give a complete answer on the question about stability, due to the presence of the complicated polynomial (4.4.16). The advantage of our general approach, however, lies in its generality. Of course, with precise specification of the functional form of the investment functions, the polynomial can be explored and explicit stability conditions can be obtained. But even in the case of general investment function, the Proposition 4.4.4*(i)* can be simplified in the case if the surviving agents satisfy a *homogeneity* requirement concerning their EWMA estimators

Corollary 4.4.3. *Let $\boldsymbol{x}^*$ be a fixed point of the deterministic skeleton of system* (4.4.1) *belonging to a $k-1$-dimensional manifold of k-survivors equilibria. If all the surviving agents possess identical values of the EWMA parameter*

$$\lambda = \lambda_1 = \cdots = \lambda_k \quad ,$$

the condition (i) in Proposition 4.4.4 becomes

$$\frac{\langle f'_y \rangle}{l'(r^*)}\,\frac{1}{r^*} < \frac{1}{1-\lambda} \quad , \qquad \frac{\langle f'_y \rangle}{l'(r^*)} < 1 \quad , \qquad \frac{\langle f'_y \rangle}{l'(r^*)}\,\frac{2+r^*}{r^*} > -\,\frac{1+\lambda}{1-\lambda} \quad , \tag{4.4.18}$$

where $\langle f'_y \rangle = \sum_{j=1}^{k} \varphi^*_j\, f'_{j,y}\big(r^*, 0\big)$.

The inequalities in (4.4.18) are identical to the stability conditions for the single-agent case, except that one has to weight the derivatives computed in the fixed point with the wealth shares of the surviving agents. Interestingly, this implies that an agent with an investment function generating unstable equilibria if present alone in the market, may survive with the same function in a stable non-generic equilibrium if other agents, with stable functions, possess, at equilibrium, high enough wealth shares.

No-arbitrage equilibria

Finally, let us consider those equilibria where $r^* = -\bar{e}$ which we identified in Proposition 4.4.2. We consider general situation and allow some agents to have zero wealth share, so that there are $k \leq N$ survivors in such equilibrium.

Proposition 4.4.5. *Let $\boldsymbol{x}^*$ be a fixed point of the deterministic skeleton of system* (4.4.1) *belonging to a $k-1$-dimensional manifold of k-survivors equilibria defined by* (4.4.4), (4.4.11) *and* (4.4.12). *Without loss of generality we can assume that first k agents survive, so that*

$$\sum_{n=1}^{k} x^*_n\, \varphi^*_n = 0 \quad , \qquad \textit{and} \qquad \varphi^*_j = 0 \qquad \forall j \in \{k+1, \ldots, N\} \quad . \tag{4.4.19}$$

If $N \geq 3$, the fixed point $\boldsymbol{x}^*$ is non hyperbolic and, consequently, non (locally) asymptotically stable. Its non-hyperbolic submanifold is the $N-2$-dimensional manifold defined in Corollary 4.4.2.

The equilibrium $\boldsymbol{x}^*$ is (locally) stable if all the roots of the following polynomial are inside the unit circle

$$\mu \prod_{j=1}^{k} (\lambda_j - \mu) + \frac{1-\mu}{\langle x^2 \rangle} \sum_{j=1}^{k} \Big(\varphi_j^* \, f'_{j,y} (1-\lambda_j) \prod_{i=1, i \neq j}^{k} (\lambda_i - \mu) \Big) \quad , \tag{4.4.20}$$

where $f'_{j,y}$ denotes the value of the partial derivative of investment function f_j with respect to the expected return y in the point $(-\bar{e}, 0)$, and

$$\langle x^2 \rangle = \sum_{n=1}^{k} \varphi_n^* \, {x_n^*}^2 \quad .$$

The equilibrium $\boldsymbol{x}^*$ is unstable if at least one of the roots of polynomial in (4.4.20) *is outside the unit circle.*

As it was the case with Proposition 4.4.4, we can characterize the stability of equilibria only through polynomial (4.4.20). For specific investment functions this polynomial can be simplified and explicit stability conditions can be derived. For instance, when all investment functions are horizontal in point $-\bar{e}$, the no-arbitrage equilibrium (if it exists) is always stable. Another example, when conditions can be written explicitly, is the case of homogeneous expectations of the surviving agents, as we show in the following

Corollary 4.4.4. *Let $\boldsymbol{x}^*$ be a no-arbitrage fixed point of the deterministic skeleton of system* (4.4.1) *belonging to a $N-2$-dimensional manifold of k-survivors equilibria. If all the surviving agents possess identical values of the EWMA parameter*

$$\lambda = \lambda_1 = \cdots = \lambda_k \quad ,$$

the condition in Proposition 4.4.5 becomes

$$\frac{\langle f'_y \rangle}{\langle x^2 \rangle} > -\frac{1}{1-\lambda} \quad , \qquad \frac{\langle f'_y \rangle}{\langle x^2 \rangle} < \frac{1}{2} \frac{1+\lambda}{1-\lambda} \quad , \tag{4.4.21}$$

where $\langle f'_y \rangle = \sum_{j=1}^{k} \varphi_j^* \, f'_{j,y}(-\bar{e}, 0)$.

It is obvious, that in the conditions of this corollary, the no-arbitrage equilibrium will be stable for sufficiently large λ.

Proposition 4.4.5 and, in particular, the last corollary, illustrates that among all no-arbitrage equilibria which lie in the $N-2$ manifold some may be stable and some not, depending on the equilibrium wealth shares of the agents. On the other hand, the motion of the system along the $N-2$ dimensional subspace consisting of the continuum of equilibria leaves the aggregate properties of the system invariant.

4.5 Examples and Applications

In the previous Sections we performed the equilibrium and stability analysis of the model with an arbitrary number of heterogeneous CRRA-type of agents operating in the market. Even if our setting is quite general, so that we, for instance, do not specify the functional form of the investment functions, the conditions for stability are obtained for all possible long-run equilibria in the model. As a consequence of generality of our framework, in some cases these conditions are presented partially in the implicit form, but in many situations they can be made explicit.

In this Section we present two examples of applications of our results. The first example is provided exclusively for illustrative purposes. In Section 4.5.1 we illustrate the scope, generality and elegance of our results by summarizing all of them in the discussion of one particular market. In Section 4.5.2 we show that the qualitative reasoning on the basis of the EML diagram can become more powerful and attractive even if compared with analytical investigations. In this example we analyze the market with investment functions derived from mean-variance optimization and discuss the effects of different agent-specific parameters on the local and global behavior of the system. This example is important, since it demonstrates that our framework is capable for the analysis of consequences of heterogeneity in expectations for the market stability, which is a common issue in the heterogeneous agent models. This discussion will be extended in Chapter 6 of the thesis.

Another relevant application of our framework, which concerns the principles according to which the market is evolving in a heterogeneous environment, will be discussed in Chapter 5 after the presentation of the more general version of the model.

4.5.1 Equilibria and their Stability on the EML Diagram

Let us review the results of the previous Sections considering an example of market in the right panel of Fig. 4.6. There are three investment strategies in such market: two horizontal, and one increasing. We denote the slope of the latter strategy as C. Analysis of this Chapter provides the following insights about the dynamics of the market with these investment strategies.

First, there are four equilibria with one survivor S_1, S_2, S_3 and U_2. In the latter equilibrium the strategy intersects the EML from below, so this equilibrium is unstable when the agent generating this equilibrium is alone in the market. Therefore, it is unstable also in many agents setting. Equilibria S_2 and S_3 are also unstable when three agents present in the market, since in each of them there exist another, more aggressive, agent. Equilibrium S_1, where only one agent with the highest horizontal strategy survives, is stable.

In this market there are also two no-arbitrage equilibria with two survivors, which are represented by the couple of points (A_1, A_2) and (A_3, A_2). In each of these equilibria the return $r^* = -\bar{e}$, wealth shares of the agents are positive, moreover, agents' (rescaled) wealths are invariant. Equilibrium (A_1, A_2) is always asymptotically stable, while stability of (A_3, A_2) can be established if the slope C of the increasing strategy is specified. Irrespectively of this specification, the latter equilibrium is stable if both agents have the same and sufficiently large EWMA weight λ.

Finally, there is a one-dimensional manifold of non-hyperbolic equilibria where all three agents survive. It can be represented by triple (A_1, A_2, A_3). The stability of these equilibria for different wealth shares φ_3^* of the agent with increasing strategy can be easily established when the slope of his strategy is specified. For instance, it is easy to see from (4.4.20) that for the stability it is sufficient that all the roots of the following quadratic polynomial

$$\mu(\lambda_3 - \mu) + (1 - \mu)\,\frac{C\,\varphi_3^*}{\langle x^2\rangle}\,(1 - \lambda_3)$$

are inside the unit circle. The stability conditions can now be derived explicitly. They are similar to (4.4.21) and read:

$$\frac{C\,\varphi_3^*}{\langle x^2\rangle} > -\frac{1}{1-\lambda_3} \quad , \qquad \frac{C\,\varphi_3^*}{\langle x^2\rangle} < \frac{1}{2}\,\frac{1+\lambda_3}{1-\lambda_3} \quad ,$$

Thus, for sufficiently large λ_3 or for sufficiently small φ_3^*, the corresponding equilibrium on the manifold is stable, though not asymptotically.

4.5.2 Heterogeneity of Expectations and Equilibrium Stability

In our general setting, the expectations of the agents may be heterogeneous only when parameter λ differs across population. On the other hand, the investment functions can be built in an arbitrary way and, therefore, can also reflect the expectations heterogeneity. We consider here the specification of the investment functions, based on the solution of the mean-variance optimization (3.2.12) with agent-specific relative risk aversion coefficient γ_n. Thus, we assume that having the expectations $E_{t-1,n}$ and $V_{t-1,n}$ about the mean and variance of future wealth $W_{t+1,n}$, agent n finds his investment share solving the following problem

$$\max_{x_{t,n}} \Big(\mathrm{E}_{t-1,n}[W_{t+1,n}] - \frac{\gamma_n}{2\,W_{t,n}}\,\mathrm{V}_{t-1,n}[W_{t+1,n}] \Big) \quad , \tag{4.5.1}$$

subject to the budget constraint (3.5.4):

$$W_{t+1,n} = W_{t,n}\,(1 - x_{t,n})\,(1 + r_f) + \frac{W_{t,n}\,x_{t,n}}{P_t}\,(P_{t+1} + D_{t+1}) \quad .$$

In this problem, which is the same as (3.2.12), $\mathrm{E}_{t-1,n}$ and $\mathrm{V}_{t-1,n}$ denote the agent's expectations about mean and variance of his future wealth $W_{t+1,n}$, and γ_n is the coefficient of the relative risk aversion. The solution of the problem is provided by (3.2.13), and in rescaled variables it reads:

$$x_{t,n} = \frac{1}{\gamma_n\,(1 + r_f)}\,\frac{E_{t-1,n}\big[r_{t+1} + e_{t+1}\big]}{V_{t-1,n}\big[r_{t+1} + e_{t+1}\big]} \quad . \tag{4.5.2}$$

Consistently with Assumptions 1 and 2′, the first two moments $\bar{e}$ and σ_e^2 of the i.i.d. yield process are known to the traders, while the expectations about return depend on the past market history through the EWMA estimators and, therefore, can be heterogeneous. More precisely, we assume the following expressions[5]

$$E_{t-1,n}\big[r_{t+1} + e_{t+1}\big] = \delta_n + d_n\,(y_{t-1,n} + \bar{e})\;, \tag{4.5.3}$$

$$V_{t-1,n}\big[r_{t+1} + e_{t+1}\big] = z_{t-1,n} + \sigma_e^2\;. \tag{4.5.4}$$

[5]This specification is analogous, but not strictly equivalent, to the specification analyzed in Chiarella and He (2001). The equivalent demand functions are derived in Chapter 6, where we demonstrate that our model can reproduce the results of the model in Chiarella and He. Our goal in this Section, instead, is the application of the results of our analysis to the investigation of the consequences in the heterogeneity in expectations. Consequently we choose the simplest possible specification.

The linear functional form in (4.5.3) allows for a simple interpretation: the parameter $\delta_n > 0$ can be interpreted as a *risk premium*, while the extrapolation parameter d_n characterizes the relation of present investment choice with past market dynamics. Using this parameter one can distinguish between different stylized types of trading behavior. A trader n with $d_n = 0$ can be identified with a *fundamentalist*, since his investment choice is unaffected by the past return realizations. For $d_n \neq 0$ the agent is a *chartist*. Specifically, he is a *trend follower* if $d_n > 0$ and a *contrarian* if $d_n < 0$. Higher values of the past returns lead to higher investment choices for trend followers and to lower investment choices for contrarians.

Plugging (4.5.3) and (4.5.4) into (4.5.2) one gets the following investment function

$$f_n(y_{t,n}, z_{t,n}) = \frac{1}{\gamma_n\,(1 + r_f)} \frac{\delta_n + d_n\,(y_{t,n} + \bar{e})}{z_{t,n} + \sigma_e^2} \quad . \tag{4.5.5}$$

This function satisfies, obviously, our Assumption $2'$. Consequently, as the results of Section 4.3 imply, the location of equilibria and their stability analysis can be performed considering the restriction of (4.5.5) to the line $z_{t,n} = 0$. The restricted investment function possesses a simple linear form

$$f_n(y_{t,n}, 0) = \tilde{\delta}_n + \tilde{d}_n\,(y_{t,n} + \bar{e})\ , \quad \text{with} \quad \tilde{\delta}_n = \frac{\delta_n}{\gamma_n\,(1 + r_f)\,\sigma_e^2}\ , \quad \tilde{d}_n = \frac{d_n}{\gamma_n\,(1 + r_f)\,\sigma_e^2}\ . \tag{4.5.6}$$

Let us now provide some examples on the use of the EML "plot" to investigate the role of the parameters of (4.5.6) in the definition of the number, location and stability of equilibria. We do not present a complete analysis, but starting from these examples it is easy to develop further the formal machinery.

In the left panel of Fig. 4.8 three investment functions (4.5.6) are shown for the agents with the same value of risk premium $\tilde{\delta} \in (0, 1)$ and different values of extrapolation parameter $\tilde{d}_n$. The horizontal investment function corresponds to $\tilde{d}_1 = 0$, i.e. to the fundamentalist behavior. The fundamentalist function has one equilibrium, denoted with F. From the picture it is obvious that varying the value of the risk premium, that is the amount of wealth invested in the risky asset, the horizontal investment function always has (apart the forbidden value $x = 1$) a unique equilibrium.

Now consider value of the extrapolation parameter $\tilde{d}_2 < 0$, associated with contrarian behavior. The equilibrium in the right branch of the EML, denoted with C_1, moves toward lower values of x^* and r^*. At the same time, a second equilibrium, denoted as C_2, appears as the intersection of the investment function with the left branch of the EML. This second equilibrium is characterized by a low value of the price return $r^* < -\bar{e}$. Notice that it belongs to the feasible part of the left branch (i.e. leads to positive equilibrium prices) only if $\tilde{d}_2$ is small enough. It follows that the contrarian investment function always possesses two equilibria, but one of them may be unfeasible.

Finally, for $\tilde{d}_3 > 0$, the agent is a trend-follower. For low enough values of $\tilde{d}_3$, his investment function possesses two intersections, denoted with T_1 and T_2, with the right branch of the EML. With the increase of the value of $\tilde{d}_3$, the equilibrium return associated with the first intersection increases while the one associated with the second intersection decreases until they coincide and then disappear, in the point where the investment function is tangent to the EML. If one considers relatively high values of risk premium $\tilde{\delta}$ and of the slope $\tilde{d}_3$, the trend follower function can also possess one or two intersections with the left branch of the EML. Thus, the trend-followers investment function possesses 0, 1 or 2 equilibria.

Concerning equilibria stability in the single agent case, from Proposition 4.3.6 we immediately see that equilibrium F of the fundamentalist is always stable, while equilibrium T_2 of

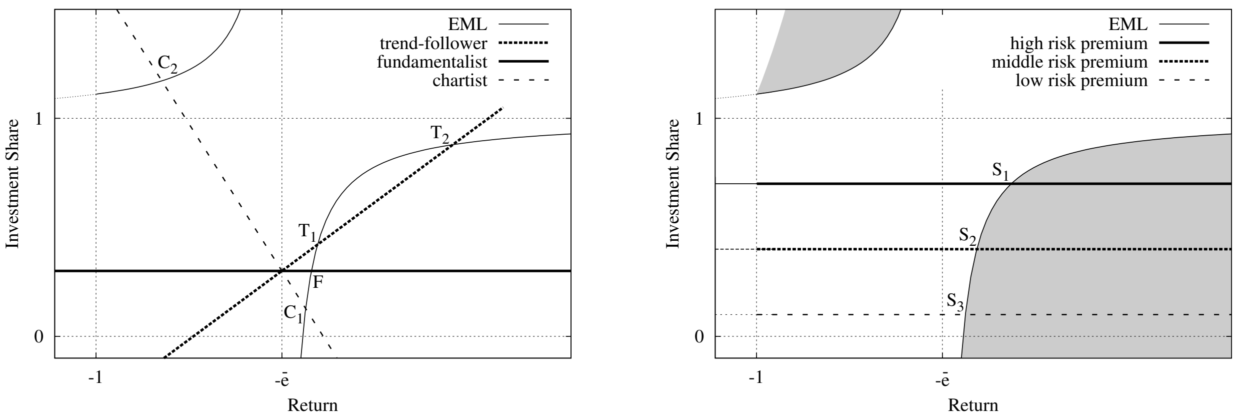

Figure 4.8: Equilibria in the market with investment functions based on the mean-variance solution of the utility-maximization. **Left panel:** Three typical agent behaviors. **Right panel:** Equilibria with three different fundamentalists. S_1 is the only stable equilibrium.

the trend-followers is always unstable, because it violates the second inequality in (4.3.10). The other types of equilibria (like T_1, C_1 and C_2 in the left panel of Fig. 4.8) become stable with high enough value of λ.

Let us now consider a model with heterogeneous agents. We confine our illustration to the case of several fundamentalists. The extension to other scenarios is straightforward. In the right panel of Fig. 4.8 we draw three fundamentalist investment functions with different values of the risk premia $\tilde{\delta}$. There are three "generic" equilibria: S_1, S_2 and S_3. It follows from Proposition 4.4.3 that only one of them is stable, namely S_1. This is the only equilibrium in which the survivor is the most aggressive trader. Thus, among different fundamentalists, it is the one with the highest risk premium who survives. Since $\tilde{\delta}$ is inversely proportional to the risk aversion parameter γ one can say that among the fundamentalists with different risk aversion coefficients the one with the smallest coefficient, *ceteris paribus*, survives and dominates the economy. This is not a peculiar feature of the fundamentalist investment function, however. In general terms, one can describe the effect of an increase (decrease) in the agent's risk aversion as a downward (upward) shift of his investment function. The shift implies that a different quantity (lower or higher) of the risky asset is held in agent's portfolio for the same level of expected price return. Because of (4.4.15) or (4.4.17) the shift of one investment function can have a strong destabilizing effect on the equilibria of the other functions, and possibly disrupt already established dominant positions. The ensuing out-of-equilibrium dynamics can eventually drive the market towards a new equilibrium in which the agent who increased his risk aversion dominates the economy. This general property seems to be in remarkable agreement with the following result suggested by a simulation model in Zschischang and Lux (2001) (p.568, 569):

> Looking more systematically at the interplay of risk aversion and memory span, it seems to us that the former is the more relevant factor, as with different [risk aversion coefficients] we frequently found a reversal in the dominance pattern: groups which were fading away before became dominant when we reduced their degree of risk aversion...
>
> ...It also appears that when adding different degrees of risk aversion, the differences of time horizons are not decisive any more, provided the time horizon is not too

short.

In the model by Zschischang and Lux the CRRA-agents have limited memory spans (time horizons) and forecast the next return as the average of past realized returns. Such equally weighted forecast setting can be considered as a particular case of another version of our model which we will present in Chapter 5. It suffice to notice, however, that larger memory spans can be approximated by larger values of λ. Therefore, the result described in the above quotation is exactly what expected according to our Proposition 4.4.3. High enough "time horizons" are needed in order to satisfy condition (4.4.14). Then, the survivor is determined solely by conditions in (4.4.15) which hold for the agent with the highest risk aversion.

The above discussion seems to suggest that the market tends to prefer "higher" investment functions. In the proposed multi-agent example the economy always ends up in the equilibrium with the highest possible return. That is, the dynamics endogenously selects the best aggregate outcome. This is not always the case, however, as we will discuss in Section 5.4.

4.6 Conclusion

In this Chapter we have presented the analysis of the model where wealth and price both evolving over time in endogenous fashion. We have shown that our model leads to the novel results concerning the characterization and the stability of equilibria in speculative pure exchange economies with heterogeneous adaptive traders. While we mainly focused on theoretical aspects, our results also provide some rigorous background to the growing literature about numerical simulations of artificial agent-based financial markets. Let us shortly review the assumption we made and the results we obtained.

We considered a simple analytical framework using a minimal number of assumptions (2 assets and Walrasian price formation). We modeled agents as speculative traders with individual demand functions proportional to their wealth (CRRA framework). We constrained the agents' portfolio choice to be a function of the past market history, expressed through EWMA estimators of average return and variance. Under prescribed but arbitrary specification about the agents' investment choices we obtained the multi-dimensional dynamical system which describes the feasible dynamics of the economy (i.e. the dynamics for which prices stay always positive). Inside this framework, we analyzed the market populated by an arbitrarily large number of heterogeneous agents. Using the powerful tool of the Equilibrium Market Line, we found all possible market equilibria, discussed their local stability conditions and the bifurcation types generated by the violation of these conditions.

It is, however, necessary to notice that our findings possess different degrees of generality. We observed, for instance, a stabilizing effect when the "memory" of the agent, i.e. the length of the past market history that he effectively takes into consideration in his expectations, is increased. One can think that such effect is due to the particular choice of estimators we made. Indeed, an analogous property has been found in Bottazzi (2002) inside a CARA framework when also EWMA estimators are used. It may happen that with different estimators, for instance constant weighted averages, this behavior would change. On the other hand, the insensitivity of equilibria location and stability on agent evaluation of endogenous risk is probably a quite robust feature. At equilibrium the price return is constant. Consequently, any consistent estimator of the variance (or any other central moment) of the conditional returns distribution has to converge to zero.

In the next Chapter we present another version of the model, in which agents base their decisions not necessary on the EWMA estimators. They, instead, transform the past information set into the future investment choice by means of the generic investment functions defined in Assumption 2 of Chapter 3. The only restriction which we impose there is that the information set has a finite dimension (while in the version developed above, the information set was infinite-dimensional). The analysis of this model will confirm the affirmation of the previous paragraph. We will also see that all our findings about the nature of the many agents equilibria, the existence of generic single survivor equilibria, non-generic many survivors non-hyperbolic equilibria, and generic many survivors no-arbitrage equilibria have general character. Consequently, we will be able to formulate the optimal selection principle governing the asymptotic market dynamics.

Chapter 5

Individual Behavior Based upon Generic Investment Functions

This Chapter is devoted to the analysis of another version of the model introduced in Chapter 3. Substantially, the model which we investigate here is *as general as possible* in the limits of the endogenous market setting with i.i.d. yield and CRRA-type of the agents behavior. Because of the generality, this Chapter can be considered as the heart of the thesis.

The agents' demand is modeled here in total generality by means of an unspecified investment functions. However, the necessity to work with the system of finite dimension leads us to assume that an information set available to the agents is finite. This is the only limitation with respect to the framework of the previous Chapter, but it implies that the previous model cannot be considered as a particular case of this one. Rather, in a rough sense, the model in Chapter 4 can be considered as one of possible limiting cases of the model presented below when the dimension of the information set goes to infinity.

This Chapter is organized in a similar way as a previous one. We start in the next Section with presentation of a complete (and short!) list of the assumptions. Then, in Section 5.2 we consider the case of homogeneous investment choices, i.e. the single agent case. We provide the analysis of existence of equilibria and their stability. Concerning the former issue, we show that the Equilibrium Market Line introduced in Definition 4.3.1 can be used also in the present framework without substantial modifications. Concerning the question of stability, the generality of the current setting does not allow us to derive stability conditions explicitly even in the simple, single agent case, as opposite to the framework of the previous Chapter. Nevertheless, we provide the stability conditions through the so-called *stability polynomial* which depends exclusively on the agent's investment function. In Section 5.3 we consider the general case of many distinct agents operating in the market. We derive the system describing the evolution of the economy, characterize all the possible equilibria and study their stability.

Section 5.4 is devoted to the discussion of the principles according to which, in a heterogeneous environment, the dynamics of the market selects the investment functions that eventually dominate the economy. We show that the stability analysis implies a local nature for the market selection process and also impossibility to define any global dominance order relation among agents. Section 5.5 concludes this Chapter.

5.1 Generic Investment Functions

Remember that the basic framework of the model which we analyze have been introduced in the second part of Chapter 3. There, in particular, we derived the dynamics generated by this model for an arbitrary dividend process and general investment behavior. In this Chapter we leave the same specification of the dividend structure as in Section 3.7, which is described by

Assumption 2. The dividend yields e_t are i.i.d. random variables obtained from a common distribution with positive support, mean value $\bar{e}$ and variance σ_e^2.

Concerning the agents behavior, we assume that each agent n has his own, so to speak, "memory time span" L_n, so that at each time step his new investment choice is determined as a function of the last L_n return realizations. Apart the requirement that it possesses first order derivatives with respect to the past price returns, we do not restrict this function in any way. Moreover, for the following discussion, L_n must be finite, but can be arbitrarily large. Thus, general Assumption 2 is slightly modified and becomes

Assumption 2″. For each agent n there exists a memory time span L_n and differentiable **investment function** f_n which maps the present information set consisting of the past L_n available returns into his investment share:

$$x_{t,n} = f_n(r_{t-1}, \ldots, r_{t-L_n}) \quad . \tag{5.1.1}$$

As usually, function f_n in the right-hand side of (5.1.1) gives a complete description of the investment decision of the n-th agent. It is defined on the set of all possible return combinations, i.e. on the set $[-1, +\infty)^{L_n}$.

Notice that our framework contains many possible specifications of the agent behavior discussed in the literature. We provide below only two examples, but in principle any two-step procedure described in Section 4.1 based on the *finite* return history can be easily incorporated in our setting. Generally speaking, agent n can forecast the distribution of the next period return as some weighted average of the past returns. Let us denote as $c_{\tau,n}$ for $\tau \in \{1, 2, \ldots, L_n\}$, the agent-specific weights of such forecast. Then the forecasts for the first two moments[1] is provided by the following statistics:

$$\begin{aligned} g_{1,n} &= \sum_{\tau=1}^{L_n} c_{\tau,n}\, r_{t-\tau} \\ g_{2,n} &= \sum_{\tau=1}^{L_n} c_{\tau,n}\, \left(r_{t-\tau} - g_{1,n}\right)^2 \end{aligned} \quad , \qquad \text{where} \qquad \sum_{\tau=1}^{L_n} c_{\tau,n} = 1\,. \tag{5.1.2}$$

The investment function f_n in this interpretation would become a composition of some another agent specific function h_n and two estimators $g_{1,n}$ and $g_{2,n}$.

Example 1. In the case of equal weights $c_{\tau,n}$ in (5.1.2), i.e. if $c_{\tau,n} = 1/L_n$ for all τ, we get the same setting as in the microscopic simulation model of Levy, Levy and Solomon or in the analytic model of Chiarella and He (2001).

[1] The generalization for any moment of the higher order is straight-forward.

Example 2. If the weights $c_{\tau,n}$ in (5.1.2) are exponentially declining in the past, so that there exist $\lambda_n \in [0, 1)$ such that

$$c_{\tau,n} = C\,\lambda_n^{\tau-1} \quad , \qquad \forall \tau \in \{1, \dots, L_n\} \quad , \qquad \text{where} \qquad C = \frac{1-\lambda_n}{1-\lambda_n^{L_n}} \quad ,$$

we get the case with finite exponentially weighted moving average (EWMA) forecast. In the limit with $L_n \to \infty$ this case leads to the setting of Chapter 4.

The dynamical system which we will analyze in this Chapter is composed by the set of equations describing the return dynamics (3.6.4), the relative wealth evolution (3.6.6), and the agents investment behavior (5.1.1). Remember from Chapter 3 that if all investment functions f_n have the compact ranges inside the interval $(0, 1)$, then the corresponding dynamics implies a positive price at each time period. This condition is not necessary, however, so that in further analysis there are no any restriction on the investment functions.

5.2 Single Agent Case

We start with the analysis of the special situation in which a single agent operates in the market. The main reason to perform this analysis rests in its relevance for the multi-agent case. This Section starts laying down the dynamics of the single agent economy as a multi-dimensional dynamical system of difference equations of first order. All possible equilibria of the system are identified and the associated characteristic polynomial, which can be used to analyze their stability, is derived.

5.2.1 Dynamical System

In the case of one single agent the dynamical system describing the market evolution can be considerably simplified since the explicit evolution of wealth shares in (3.6.5) is not needed. As a consequence, the whole system can be described with only $L+1$ variables: one variable represents the current investment choice x_t and the other variables the L past returns[2].

The current return can be defined by means of the function in the right hand-side of (3.6.4):

$$R(x', x, e) = \frac{x' - x + e\,x'\,x}{(1-x')\,x} \quad , \tag{5.2.1}$$

where e and x' denote the current (contemporaneous with return) dividend yield and investment choice, respectively, while x is the investment choice of the previous period.

With such definitions the dynamical system governing the evolution of the economy with a single agent reads

$$\begin{cases} x_{t+1} & = & f\big(r_{t,0}, r_{t,1}, \dots, r_{t,L-1}\big) \\ r_{t+1,0} & = & R\Big(f\big(r_{t,0}, r_{t,1}, \dots, r_{t,L-1}\big),\; x_t,\; e_{t+1}\Big) \\ r_{t+1,1} & = & r_{t,0} \\ & \vdots & \\ r_{t+1,L-1} & = & r_{t,L-2} \end{cases} \quad , \tag{5.2.2}$$

[2]Since only one agent is present in the market, we omit index 1 from any agent-specific variable.

where $r_{t,l}$ stands for the price return at time $t-l$.

In the rest of this Section we are interested in analyzing the so-called *deterministic skeleton* of this $L+1$-dimensional system, so that we substitute the yield by its mean value $\bar{e}$.

5.2.2 Determination of Equilibria

In the following analysis, in order to give a simple geometric characterization of equilibria of system (5.2.2) we will use the notion of Equilibrium Market Line which was introduced in the previous Chapter. We repeat this definition below

Definition 5.2.1. The *Equilibrium Market Line* (EML) is the function $l(r)$ defined according to

$$l(r) = \frac{r}{\bar{e}+r} \quad . \tag{5.2.3}$$

Let x^* denotes the agent's wealth share invested in the risky asset at equilibrium and let r^* be the the equilibrium return. In any fixed point the realized returns are constant, so that $r_0 = r_1 = \cdots = r_{L-1} = r^*$. One has the following

Proposition 5.2.1. *Let* $\boldsymbol{x^*} = (x^*; r^*, \ldots, r^*)$ *be a fixed point of system* (5.2.2)*. Then it is:*

(i) The equilibrium return r^* and the equilibrium investment share x^* satisfy

$$l(r^*) = f\big(\underbrace{r^*, \ldots, r^*}_{L}\big)\,, \qquad x^* = f\big(\underbrace{r^*, \ldots, r^*}_{L}\big) \quad . \tag{5.2.4}$$

(ii) The point $\boldsymbol{x^*}$ is a feasible equilibrium, i.e. the equilibrium prices are positive, if either $x^* < 1$ or $x^* \geq 1/(1-\bar{e})$.

(iii) The equilibrium growth rate of the agent's wealth is equal to price return r^*.

Proof. See appendix E.2. □

This statement is completely analogous to Proposition 4.3.1 and coincides with it in the case $L = 1$. Proposition 5.2.1*(i)* implies that the characterization of all equilibria can be obtained by means of the Equilibrium Market Line from Definition 5.2.1. Indeed, all equilibria of system (5.2.2) can be found as the intersections of the EML with the full symmetrization of function f, i.e. with the restriction of this function to the one dimensional subspace defined by $r_0 = r_1 = \cdots = r_L$. Notice that if the agent's investment decision is based upon the return statistics in (5.1.2), then after symmetrization we get $g_1 = r^*$ and $g_2 = 0$. Therefore, above statement generalizes Proposition 4.3.5 on the case of arbitrary forecasting behavior based upon finite return history.

Proposition 5.2.1*(ii)* states that economically meaningful equilibria are characterized by values of the investment share inside the intervals $(-\infty, 1)$ or $[1/(1-\bar{e}), +\infty)$. This is equivalent to the restriction $r^* \geq -1$. Finally, according to the last part of Proposition 5.2.1, the growth rate of the agent's wealth coincides with the price growth rate.

As a first example of the application of Proposition 5.2.1, let us consider investment functions which are one-dimensional functions of the sole last return (i.e. $L = 1$). In Fig. 5.1, which repeats Fig. 4.1, two functions of this type are shown (thick lines) together with the hyperbolic curve representing the EML defined in (5.2.3) (thin line). The intersections of the

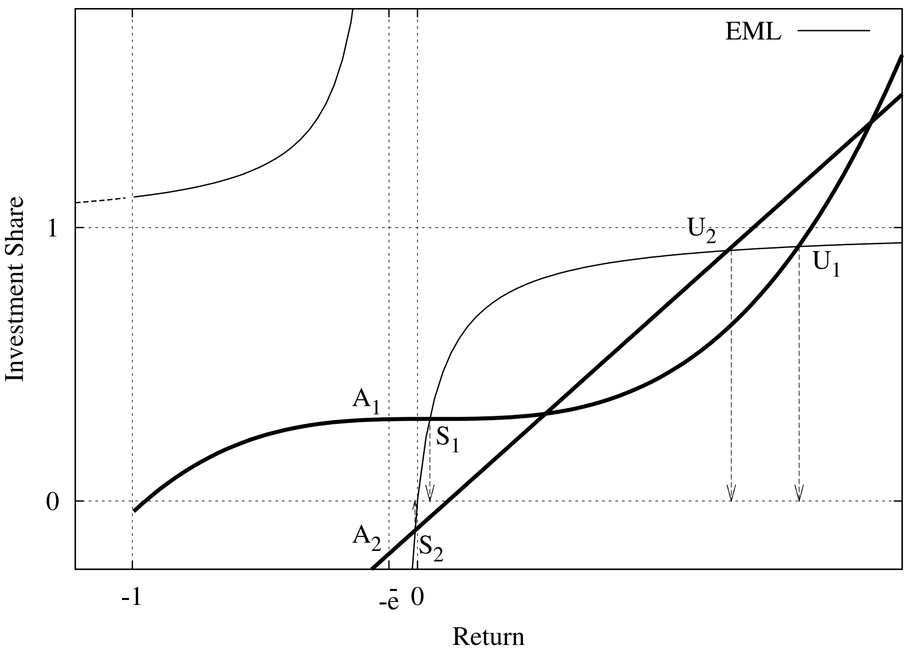

Figure 5.1: Investment functions (thick lines) based on the last realized return. The equilibria are found as intersections with the EML (thin line). Both functions have two equilibria: S_1 and U_1 the non-linear, S_2 and U_2 the linear.

investment function with the EML are the possible equilibria of the system. The ordinate of the intersection gives the value of equilibrium investment share x^*, while the abscissa gives the equilibrium return r^*.

We also remind that there are three qualitatively different scenarios[3]. In equilibria with $r^* \in [-1, -\bar{e})$ the investment in the risky asset is characterized by negative gross (rescaled) return $r^* + \bar{e} < 0$, the agent maintains a long position in the risky asset ($x^* > 1$) and has negative wealth return. In equilibria with $r^* \in (-\bar{e}, 0)$ the capital gain on the risky asset in terms of unscaled price is negative, nevertheless the gross return $r^* + \bar{e}$ is positive due to the dividend yield. The agent maintains a short position in the risky asset ($x^* < 0$) and his wealth return is again negative. Finally, if $r^* \in (0, +\infty)$ the price return is positive, the agent position is characterized by a fixed fraction of wealth invested in the risky asset $x^* \in (0, 1)$ and his wealth return is positive. Fourth scenario with $r^* = -\bar{e}$ is impossible for the single agent case.

A second geometric example is presented in Fig. 5.2 and corresponds to $L = 2$. The two-dimensional surface represents the investment function depending on the two last realized returns, r_0 and r_1. Here r_0 stands for the last period return while r_1 is the return of the period before the last. The thick line on the function surface is the intersection of the investment function with the "symmetric" plane defined by the condition $r_0 = r_1$. On the same plane the curve relative to EML $l(r)$ is also drawn. The intersections of these two curves define all possible equilibria. In Fig. 5.2 there is one trivial equilibrium with zero return and a second equilibrium with positive price return r^* and equilibrium investment share $x^* = f(r^*, r^*)$.

The same analysis can be applied, unmodified, to the investment functions with higher values of L. The bottom line of these examples and of the previous discussion is that the

[3]Remember that the analysis is performed with respect to the rescaled variables as defined in (3.5.7).

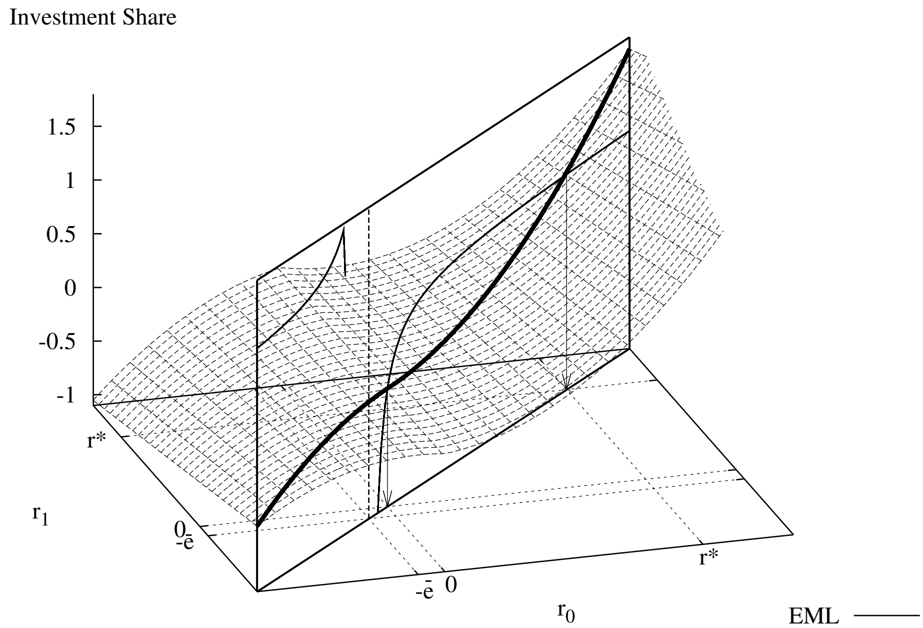

Figure 5.2: Investment function based on the last two realized returns $f(r_{t-1}, r_{t-2})$ and its intersection (thick line) with the plane $r_{t-1} = r_{t-2}$. Equilibria are found on this plane as intersections with the EML (thin line).

agent's memory span L is irrelevant for the question of the existence and location of equilibria: only the restriction of the investment function f on the "symmetric" plane is relevant and, in all cases, the equilibria are located on the one dimensional EML and can be presented in a diagram analogous to Fig. 5.1.

5.2.3 Stability Conditions of Equilibria

In this Section we discuss the stability conditions of the equilibria that have been identified in the previous Section. We derive these conditions from the analysis of the roots of the characteristic polynomial associated with the Jacobian of system (5.2.2) computed at equilibrium. The characteristic polynomial does, in general, depend on the behavior of the individual investment function f in an infinitesimal neighborhood of the equilibrium $\boldsymbol{x}^*$. This dependence can be summarized with the help of the following

Definition 5.2.2. The *stability polynomial* $P(\mu)$ of the investment function f in $\boldsymbol{x}^*$ is

$$P_f(\mu) = \frac{\partial f}{\partial r_0}\,\mu^{L-1} + \frac{\partial f}{\partial r_1}\,\mu^{L-2} + \cdots + \frac{\partial f}{\partial r_{L-2}}\,\mu + \frac{\partial f}{\partial r_{L-1}}\ , \tag{5.2.5}$$

where all the derivatives of f are computed in point $\big(\underbrace{r^*, \ldots, r^*}_{L}\big)$.

Using the previous definition, the equilibrium stability conditions can be formulated in terms of the equilibrium return r^*, and of the slope of the Equilibrium Market Line in equilibrium

$$l'(r^*) = \frac{\bar{e}}{(\bar{e}+r^*)^2} \quad . \tag{5.2.6}$$

The following applies

Proposition 5.2.2. *The fixed point $\boldsymbol{x^*} = (x^*; r^*, \ldots, r^*)$ of system* (5.2.2) *is (locally) asymptotically stable if all the roots of the polynomial*

$$Q(\mu) = \mu^{L+1} - \frac{P_f(\mu)}{r^*\, l'(r^*)} \Big(\big(1+r^*\big)\, \mu - 1 \Big) \quad , \tag{5.2.7}$$

are inside the unit circle.

The equilibrium $\boldsymbol{x^*}$ is unstable if at least one of the roots of $Q(\mu)$ lies outside the unit circle.

Proof. See appendix E.3. □

Once investment function f is known, polynomial $P_f(\mu)$ and, in turn, polynomial $Q(\mu)$ can be explicitly derived. In some cases, stability conditions can be provided explicitly. For instance, when $L = 1$ the polynomial $Q(\mu)$ has second degree. Then, applying the results of Appendix A, the stability conditions can be derived. It is not surprising that they coincide with conditions (4.3.4) in Proposition 4.3.2 where we analyzed the "naïve forecast" specification of the agent investment behavior. Of course, such forecasting interpretation is not needed in the present framework, which is based upon general Assumption 2″.

The analysis of the roots of $Q(\mu)$ can be used to reveal the role of the different parameters in stabilizing or destabilizing a given equilibrium. In particular, one can prove that when L goes to infinity in Example 2 of Section 5.1, i.e. for the exponentially weighted moving average forecast, then the roots of polynomial $Q(\mu)$ are such that the stability region is described by inequalities in (4.3.7). Therefore, in the case when agent uses EWMA estimators any point can be stabilized, when $\lambda \to 1$ and $L \to \infty$. The numerical investigation of the roots of the corresponding polynomial shows that this is not the case for an infinite increase of L in the Example 1 of Section 5.1 with constant weighted average forecast.

5.3 Economy with Many Agents

This Section extends the previous results to the case of a finite, but arbitrarily large, number of heterogenous agents. Each agent n possesses his own investment function f_n based on a finite number L_n of past market realizations. Without loss of generality we can assume that the memory spans of different functions f_n are all the same and equal to the largest span $L = \max\{L_1, \ldots L_N\}$, so that each investment function can be thought as having exactly L arguments

$$x_{t+1,n} = f_n(r_t, r_{t-1}, \ldots, r_{t-L+1}) \; . \tag{5.3.1}$$

This Section is organized as the previous one. It starts with the derivation of the $2N + L - 1$ dimensional stochastic dynamical system which describes the evolution of the economy and continues with an identification of all possible equilibria of the associated deterministic skeleton and the analysis of their stability.

5.3.1 Dynamical System

If there is more than one agent in the market, the evolution of agents wealth is not decoupled from the system and, consequently, all equations (3.6.6) are relevant for the dynamics. The first-order dynamical system associated with (3.6.4) and (3.6.6) with investment functions in (5.3.1) is defined in terms of the following $2N+L-1$ independent variables

$$x_{t,n} \quad \forall n \in \{1,\dots,N\}\,; \quad \varphi_{t,n} \quad \forall n \in \{1,\dots,N-1\}\,; \quad r_{t,l} \quad \forall l \in \{0,\dots,L-1\}\,, \tag{5.3.2}$$

where $r_{t,l}$ denotes the price return at time $t-l$. Notice that only $N-1$ wealth shares are needed. Indeed, at any time step t, it is $\sum_{n=1}^{N} \varphi_{t,n} = 1$ so that $\varphi_{t,N} = 1 - \sum_{n=1}^{N-1} \varphi_{t,n}$. The dynamics of the system is provided by the following

Lemma 5.3.1. *The* $2N+L-1$ *dynamical system defined by* (3.6.4) *and* (3.6.6) *in terms of the variables in* (5.3.2) *reads*

$$\begin{array}{l}
\mathcal{X}: \left[\begin{array}{lcl}
x_{t+1,1} & = & f_1\big(r_{t,0},\dots,r_{t,L-1}\big) \\
\vdots & \vdots & \vdots \\
x_{t+1,N} & = & f_N\big(r_{t,0},\dots,r_{t,L-1}\big)
\end{array}\right. \\
\mathcal{W}: \left[\begin{array}{lcl}
\varphi_{t+1,1} & = & \Phi_1\Big(x_{t,1},\dots,x_{t,N};\varphi_{t,1},\dots,\varphi_{t,N-1};e_{t+1}; \\
 & & \quad R\big(f_1(r_{t,0},\dots,r_{t,L-1}),\dots,f_N(r_{t,0},\dots,r_{t,L-1}); \\
 & & \quad\quad x_{t,1},\dots,x_{t,1};\varphi_{t,1},\dots,\varphi_{t,N-1};e_{t+1}\big)\Big) \\
\vdots & \vdots & \vdots \\
\varphi_{t+1,N-1} & = & \Phi_{t,N-1}\Big(x_{t,1},\dots,x_{t,N};\varphi_{t,1},\dots,\varphi_{t,N-1};e_{t+1}; \\
 & & \quad R\big(f_1(r_{t,0},\dots,r_{t,L-1}),\dots,f_N(r_{t,0},\dots,r_{t,L-1}); \\
 & & \quad x_{t,1},\dots,x_{t,N};\varphi_{t,1},\dots,\varphi_{t,N-1};e_{t+1}\big)\Big)
\end{array}\right. \\
\mathcal{R}: \left[\begin{array}{lcl}
r_{t+1,0} & = & R\Big(f_1(r_{t,0},\dots,r_{t,L-1}),\dots,f_N(r_{t,0},\dots,r_{t,L-1}); \\
 & & \quad x_{t,1},\dots,x_{t,N};\varphi_{t,1},\dots,\varphi_{t,N-1};e_{t+1}\Big) \\
r_{t+1,1} & = & r_{t,0} \\
\vdots & \vdots & \vdots \\
r_{t+1,L-1} & = & r_{t,L-2}
\end{array}\right.
\end{array} \quad , \tag{5.3.3}$$

where

$$\begin{aligned}
&R\Big(y_1,y_2,\dots,y_N;x_1,x_2,\dots,x_N;\varphi_1,\varphi_2,\dots,\varphi_{N-1};e\Big) = \\
&= \frac{\sum_{n=1}^{N-1} \varphi_n\big(y_n\,(1+e\,x_n)-x_n\big) + \Big(1-\sum_{n=1}^{N-1}\varphi_n\Big)\,\big(y_N\,(1+e\,x_N)-x_N\big)}{\sum_{n=1}^{N-1}\varphi_n\,x_n\,(1-y_n) + (1-\sum_{n=1}^{N-1}\varphi_n)\,x_N\,(1-y_N)}
\end{aligned} \tag{5.3.4}$$

and

$$\begin{aligned}
&\Phi_n\big(x_1,x_2,\dots,x_N;\varphi_1,\varphi_2,\dots,\varphi_{N-1};e;R\big) = \\
&= \varphi_n\,\frac{1+x_n\,(R+e)}{1+(R+e)\,\Big(\sum_{m=1}^{N-1}\varphi_m\,x_m + \big(1-\sum_{m=1}^{N-1}\varphi_m\big)\,x_N\Big)} \qquad \forall n \in \{1,\dots,N-1\}\,.
\end{aligned} \tag{5.3.5}$$

Proof. We ordered the equations to obtain three separated blocks: $\mathcal{X}$, $\mathcal{W}$ and $\mathcal{R}$. In block $\mathcal{X}$ there are N equations defining the investment choices of agents. Block $\mathcal{W}$ contains $N-1$ equations describing the evolution of the wealth shares. Finally, block $\mathcal{R}$ is composed by L equations which describe the evolution of the return. In the last block equations are in ascending order with respect to the time lag.

The set $\mathcal{X}$ is immediately obtained from the definition of the investment functions (5.3.1). The first equation of block $\mathcal{R}$ is (3.6.4) rewritten in terms of variables (5.3.2) using (5.3.4) and (3.6.2), while the remaining equations are just the result of a "lag" operation. Notice that (5.3.4) reduces to (5.2.1) in the case of a single agent. Finally, the evolution of wealth shares described in block $\mathcal{W}$ is obtained from (3.6.6) once the notation introduced in (3.6.2) is explicitly expanded. Notice that, due to the presence of function R in the last expression, all functions Φ_n depend on the same set of variables as R. □

The rest of this Section is devoted to the analysis of the *deterministic skeleton* of (5.3.3): we replace the yield realizations $\{e_t\}$ by their mean value $\bar{e}$ and analyze the equilibria of the resulting deterministic system.

5.3.2 Determination of Equilibria

The characterization of fixed points of system (5.3.3) is in many respects similar to the system in the previous Chapter. In particular, we identify two types of equilibria: "EML" equilibria which, as in the single agent case, can be described through the hyperbolic Equilibrium Market Line, and "no-arbitrage" equilibria where $r^* = -\bar{e}$. Let

$$\boldsymbol{x}^* = \big(x_1^*, \ldots, x_N^*; \varphi_1^*, \ldots, \varphi_{N-1}^*; \underbrace{r^*, \ldots, r^*}_{L}\big)$$

denote a fixed point where r^* is the equilibrium return and x_n^* and φ_n^* denote the equilibrium values of the investment function and the equilibrium wealth share of agent n, respectively. In this notation we remind the following

Definition 5.3.1. Agent n is said to *"survive"* in $\boldsymbol{x}^*$ if his equilibrium wealth share is strictly positive, $\varphi_n^* > 0$. Agent n is said to *"dominate"* agent n' in $\boldsymbol{x}^*$ if $\varphi_{n'}^*/\varphi_n^* = 0$. An agent n who dominates, at equilibrium, any other agent $n' \neq n$ is said to *"dominate"* the economy.

EML equilibria

We start the identification of all equilibria from those which generalize the equilibria for the single agent case. The following statement characterizes all EML equilibria of system (5.3.3).

Proposition 5.3.1. *Let $\boldsymbol{x}^*$ be a fixed point of the deterministic skeleton of system* (5.3.3) *with $r^* \neq -\bar{e}$. Two mutually exclusive cases are possible:*

(i) ***Single agent survival.*** *In $\boldsymbol{x}^*$ only one agent survives and, therefore, dominates the economy. Without loss of generality we can assume this agent to be agent* 1 *so that for the equilibrium wealth shares one has*

$$\varphi_n^* = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \quad . \tag{5.3.6}$$

The equilibrium return r^* is determined as the solution of

$$l(r^*) = f_1(r^*, \dots, r^*) \quad , \tag{5.3.7}$$

while the equilibrium investment shares are defined according to

$$x_n^* = f_n(r^*, \dots, r^*) \qquad \forall n \in \{1, \dots, N\}. \tag{5.3.8}$$

The wealth growth rate of the survivor at equilibrium is equal to the equilibrium price return, $\rho_1^* = r^*$.

(ii) ***Many agents survival.*** *In $\boldsymbol{x}^*$ more than one agent survives. Without loss of generality one can assume that the agents with non-zero wealth shares are the first k agents (with $k > 1$) so that the equilibrium wealth shares satisfy*

$$\begin{cases} \varphi_n^* \in (0,1) & \text{if } n \le k \ , \\ \varphi_n^* = 0 & \text{if } n > k \end{cases} \quad , \qquad \sum_{n=1}^{k} \varphi_n^* = 1 \quad . \tag{5.3.9}$$

The first k agents possess the same investment share $x_{1\diamond k}^*$ at equilibrium

$$x_1^* = x_2^* = \dots = x_k^* = x_{1\diamond k} \quad . \tag{5.3.10}$$

and equilibrium return r^* must simultaneously satisfy the following set of k equations

$$l(r^*) = f_n(r^*, \dots, r^*) = x_{1\diamond k}^* \quad \forall n \in \{1, \dots, k\} \quad . \tag{5.3.11}$$

The equilibrium investment shares of the last $N-k$ agents are defined according to

$$x_n^* = f_n(r^*, \dots, r^*) \quad \forall n \in \{k+1, \dots, N\} \quad . \tag{5.3.12}$$

The wealth growth rates of the survivors at equilibrium are equal to the equilibrium price return, i.e. $\rho_n^* = r^*$ for $n \le k$.

Proof. See appendix E.4. □

This Proposition is a direct analogy of Proposition 4.4.1. To repeat, the difference between two situations described in Proposition 5.3.1*(i)* and Proposition 5.3.1*(ii)* is two-fold.

The first aspect concerns the geometrical nature of the locus of equilibria. If in item *(i)* a precise value for each component (x^*, φ^* and r^*) of the equilibrium $\boldsymbol{x}^*$ is defined, and, therefore, a single point is uniquely determined, in item *(ii)* a hyperplane of equilibria is identified. In this case there is a residual degree of freedom in the definition of the equilibrium: while r^* and investment shares x^*'s are uniquely defined, the only requirement on the equilibrium wealth shares of the surviving agents is the fulfillment of the second equality in (5.3.9). Consequently, the following statement analogous to Corollary 4.4.1 can be immediately formulated

Corollary 5.3.1. *Consider the deterministic skeleton of system* (5.3.3)*. If it possesses an equilibrium $\boldsymbol{x}^*$ with k survivors it possesses a $k-1$-simplex of k-survivors equilibria constituted by all the points obtained from $\boldsymbol{x}^*$ through a change in the relative wealths of the survivors.*

If the survivors are the first k agents as in (5.3.9), this set can be written as

$$\left\{ \left(x_1^*, \ldots, x_N^*; a_1, \ldots, a_k, \underbrace{0, \ldots, 0}_{N-1-k}; \underbrace{r^*, \ldots, r^*}_{L} \right) \quad \middle| \quad \sum_{j=1}^{k} a_j = 1 \ , \quad a_j > 0 \right\}$$

The second difference lies in the fact that many survivors equilibria are non-generic, since they exist only in the particular case in which the k investment functions $f_1, \ldots, f_k$ have a common intersection with the EML. We illustrated this point in Fig. 4.5.

Proposition 5.3.1 shows that the link between multi-agent equilibria and single-agent equilibria is essentially the same as the corresponding link in the model of Chapter 4. Namely, the determination of the equilibrium return level r^* for the multi-agent case in (5.3.7) or (5.3.11) is identical to the case where the agent, or one of the agents, who would survive in the multi-agent equilibrium, is present alone in the market. Therefore, the geometrical interpretation of market equilibria presented in Section 5.2.2 can be extended to illustrate how equilibria with many agents are determined. For instance, there are four possible equilibria in the market with two agents in Fig. 5.1. In two of them (S_1 and U_1) the first agent, with non-linear investment function, survives such that $\varphi_1^* = 1$. In the other two equilibria (S_2 and U_2) the second agent, with linear investment function, survives so that, in these points, $\varphi_1^* = 0$.

No-arbitrage equilibria

In the equilibria identified so far, the return cannot be equal to $-\bar{e}$. However, no-arbitrage equilibria are also possible. From system (5.3.3) and definition of function R in (5.3.4) it is straight-forward to see that the following applies

Proposition 5.3.2. *Let $\boldsymbol{x}^*$ be a fixed point of the deterministic skeleton of system* (5.3.3) *with $r^* = -\bar{e}$. Then, it is*

$$\sum_{n=1}^{N} x_n^* \, \varphi_n^* = 0 \quad , \tag{5.3.13}$$

where equilibrium investment shares are defined as

$$x_n^* = f_n\big(\underbrace{-\bar{e}, \ldots, -\bar{e}}_{L}\big) \qquad \forall n \in \{1, \ldots, N\} \quad . \tag{5.3.14}$$

The wealth growth rates of all agents are zero in such equilibria.

This Proposition is analogous to Proposition 4.4.2. It describes those equilibria with many survivors which are "generic", as opposite to the situation described in Proposition 5.3.1*(ii)*. These equilibria do not exist if all agents have the investment shares of the same sign. In particular, in the same way as Corollary 4.4.2, one can prove the following

Corollary 5.3.2. *Consider the deterministic skeleton of system* (5.3.3)*. This skeleton possesses no-arbitrage equilibrium $\boldsymbol{x}^*$ with $r^* = -\bar{e}$* if and only if *there are two agents i and j whose investment functions satisfy to the following condition*

$$f_i\big(\underbrace{-\bar{e}, \ldots, -\bar{e}}_{L_i}\big) \; f_j\big(\underbrace{-\bar{e}, \ldots, -\bar{e}}_{L_j}\big) < 0 \quad . \tag{5.3.15}$$

If this condition is satisfied and $N = 2$, then such equilibrium is unique. If this condition is satisfied and $N > 2$, then the skeleton possesses the following $N-2$-dimensional manyfold of many survivors equilibria

$$\left\{ \Big(x_1^*, \ldots, x_N^*; a_1, \ldots, a_{N-1-k}; \underbrace{-\bar{e}, \ldots, -\bar{e}}_{L} \Big) \quad \Bigg| \quad \sum_{j=1}^{N} a_j = 1 \,, \sum_{j=1}^{N} a_j x_j^* = 0 \,, a_j \geq 0 \right\}$$

constituted by all the points obtained from $\boldsymbol{x}^*$ through a change in the wealth shares of the agents in the direction orthogonal to the vector of the agents' investment shares.

The illustration of the no-arbitrage equilibria remains the same. For example, in the market depicted in Fig. 5.1 in addition to four EML equilibria, there is another equilibrium with $r^* = -\bar{e}$ which geometrically can be represented by two points A_1 and A_2 showing the corresponding investment shares of the agents. Another example we presented in Chapter 4 in the right panel of Fig. 4.6.

5.3.3 Stability Conditions of Equilibria

This Section presents three Propositions which all together provide a complete stability analysis of the equilibria of system (5.3.3). The first Proposition provides the stability condition for the generic case of one single survivor. The stability of the EML equilibria in non-generic case of many survivors is addressed in the second Proposition, where the destabilizing effect of the existence of an entire hyperplane of equilibria is revealed. Finally, the third Proposition deals with the no-arbitrage equilibria identified in Proposition 5.3.2. The proofs of these Propositions require quite cumbersome algebraic manipulations and, unfortunately, are not identical (even if similar) to the proofs of the corresponding Propositions 4.4.3, 4.4.4 and 4.4.5. Thus, we provide below only the statements of the Propositions and refer the reader to Appendix E.5 for the intermediate Lemmas and the final proofs. The discussion concerning economic interpretation of these Propositions and analysis of their consequences for the aggregate behavior of the system are postponed to the next Section.

Equilibrium with single survivor

For the generic case of a single survivor equilibrium we have the following

Proposition 5.3.3. *Let $\boldsymbol{x}^*$ be a fixed point of* (5.3.3) *associated with a single survivor equilibrium. Without loss of generality we can assume that the survivor is the first agent, so that*

$$\varphi_n^* = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \quad .$$

Denote with $P_{f_1}(\mu)$ the $(L-1)$-dimensional stability polynomial associated with the investment function of the first agent f_1. With the above hypothesis, point $\boldsymbol{x}^*$ is (locally) asymptotically stable if the two following conditions are met:

1) *all the roots of polynomial*

$$Q_1(\mu) = \mu^{L+1} - \frac{(1 + r^*)\,\mu - 1}{r^*\, l'(r^*)} P_{f_1}(\mu) \quad , \tag{5.3.16}$$

are inside the unit circle.

2) *the equilibrium investment shares of the non-surviving agents satisfy the following relations*

$$-2 - r^* < x_n^* \left(r^* + \bar{e}\right) < r^* \,, \quad 1 < n \leq N \,. \tag{5.3.17}$$

The equilibrium $\boldsymbol{x}^*$ *is unstable if at least one of the roots of polynomial in* (5.3.16) *is outside the unit circle or if at least one of the inequalities in* (5.3.17) *holds with the opposite (strict) sign.*

In particular, the system exhibits a fold *bifurcation if one of the* $N-1$ *right-hand inequalities in* (5.3.17) *becomes an equality and a* flip *bifurcation if one of the* $N-1$ *left-hand inequalities becomes an equality.*

One can see that this statement as a generalization of Proposition 4.4.3. Indeed, we again found that the stability condition for a generic fixed point in the multi-agent economies is twofold. First, comparison between $Q_1(\mu)$ and polynomial $Q(\mu)$ from (5.2.7) implies that equilibrium should be "self-consistent", i.e. it should remain stable if any non-surviving agent is removed from the economy. In addition, two inequalities in (5.3.17) should hold. They coincide with (4.4.15) and were illustrated in Fig. 4.7. We stress again that these conditions imply that in those equilibria where the economy grows, the surviving agent must be the most aggressive, while in those equilibria where the overall wealth of the economy shrinks, the survivor has to be the least aggressive.

The geometric illustration of the previous Proposition is based upon the EML and was discussed in Section 4.4.3. In the situation in Fig. 5.1, if both agents present in the market at the same time, the dynamics cannot be attracted by U_1 or U_2, since these equilibria are unstable in the respective single-agent cases. Furthermore, if S_1 and S_2 are stable equilibria when the first and second function, respectively, are present alone in the market, then from (5.3.17) it follows that S_1 is the only stable equilibrium of the system with two agents.

EML equilibria with many survivors

Let us move now to consider the non-generic case, when k different agents survive in the equilibrium with $r^* \neq -\bar{e}$. The following Proposition is analogous to Proposition 4.4.4

Proposition 5.3.4. *A fixed point* $\boldsymbol{x}^*$ *of* (5.3.3) *belonging to a* $k-1$*-dimensional manifold of* k*-survivors equilibria defined by* (5.3.9), (5.3.11) *and* (5.3.12) *is never hyperbolic. Consequently,* $\boldsymbol{x}^*$ *is never (locally) asymptotically stable. Its non-hyperbolic submanifold is the* $k-1$*-simplex defined in Corollary 5.3.1.*

Let $P_{f_n}(\mu)$ *be the stability polynomial of investment function* f_n*. The equilibrium* $\boldsymbol{x}^*$ *is (locally) stable if the two following conditions are met:*

1) *all the roots of polynomial*

$$Q_{1\diamond k}(\mu) = \mu^{L+1} - \frac{(1+r^*)\,\mu - 1}{r^*\, l'(r^*)} \sum_{n=1}^{k} \varphi_n^* \, P_{f_n}(\mu) \quad , \tag{5.3.18}$$

are inside the unit circle.

2) *the equilibrium investment shares of the non-surviving agents satisfy to the following relations*

$$-2 - r^* < x_n^* \left(r^* + \bar{e}\right) < r^* \,, \quad k < n \leq N \,. \tag{5.3.19}$$

The equilibrium $\boldsymbol{x}^*$ is unstable if at least one of the roots of polynomial in (5.3.18) *is outside the unit circle or if at least one of the inequalities in* (5.3.19) *holds with the opposite (strict) sign.*

The discussion of the last Proposition essentially repeats the discussion after Proposition 4.4.4. The non-hyperbolic nature of the equilibria with many survivors is a direct consequence of their not unique specification. All equilibria belonging to the corresponding manifold are equivalent in the sense that they generate the dynamics with identical aggregate properties. However, it may be the case that some of these equilibria are stable, while others are not. This fact follows from the stability conditions provided by Proposition 5.3.4. Polynomial $Q_{1\diamond k}(\mu)$ is similar to the corresponding polynomial in Proposition 5.3.3, except that one has to weight the stability polynomial of the different investment functions $P_{f_k}(\mu)$ with the weights corresponding to the relative wealth of survivors in the equilibrium. At the same time, the constraint on the investment shares in (5.3.19) is identical to the one obtained in (5.3.17).

No-arbitrage equilibria

Finally, let us consider those equilibria where $r^* = -\bar{e}$. Analogously to Proposition 4.4.4 we consider general situation and assume that some agents have zero wealth share, so that there are $k \leq N$ survivors in such equilibrium.

Proposition 5.3.5. *Let $\boldsymbol{x}^*$ be a fixed point of the deterministic skeleton of system* (5.3.3) *belonging to a $k-1$-dimensional manifold of k-survivors equilibria defined by* (5.3.13) *and* (5.3.14)*. Without loss of generality we can assume that first k agents survive, so that*

$$\sum_{n=1}^{k} x_n^*\,\varphi_n^* = 0 \quad , \qquad \textit{and} \qquad \varphi_j^* = 0 \qquad \forall j \in \{k+1, \dots, N\} \quad . \tag{5.3.20}$$

If $N \geq 3$, the fixed point $\boldsymbol{x}^*$ is non-hyperbolic and, consequently, is not (locally) asymptotically stable. Its non-hyperbolic submanifold is the $N-2$-dimensional manifold defined in Corollary 5.3.2.

The equilibrium $\boldsymbol{x}^*$ is (locally) stable if all the roots of the following polynomial are inside the unit circle

$$\mu^{L+1} + \frac{\mu - 1}{\langle x^2 \rangle} \sum_{j=1}^{k} \varphi_j^*\, P_{f_j}(\mu) \quad , \tag{5.3.21}$$

where $P_{f_j}(\mu)$ is the stability polynomial of investment function f_j computed in point $(-\bar{e}, \dots, -\bar{e})$, and

$$\langle x^2 \rangle = \sum_{n=1}^{k} \varphi_n^*\, {x_n^*}^2 \quad .$$

The equilibrium $\boldsymbol{x}^*$ is unstable if at least one of the roots of polynomial in (5.3.21) *is outside the unit circle.*

As it was the case with Proposition 5.3.4, we can characterize the stability of equilibria only through the general polynomial (5.3.21). This polynomial is composed from the stability polynomials of individual agents introduced in Definition 5.2.2, which have to be weighted with

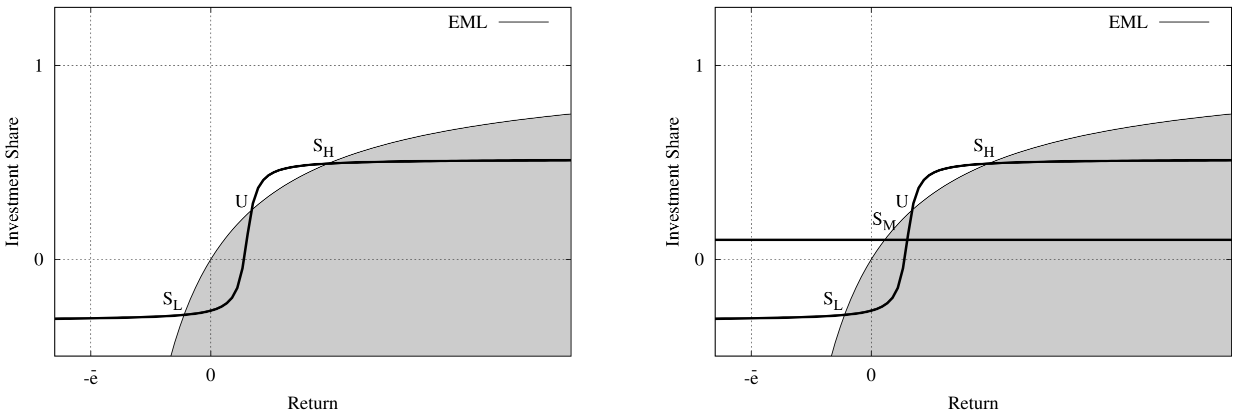

Figure 5.3: **Left panel:** Non-linear investment function leading to multiple equilibria. S_H and S_L are stable while U is unstable. **Right panel:** The same non-linear function together with a constant investment function. S_H and S_M are stable while S_L and U are unstable.

the wealth shares of the agents. Consequently, even if all no-arbitrage equilibria belonging to the $N-2$ manifold generate the dynamics which is completely equivalent on the aggregate level, the stability of any such equilibrium crucially depends on the wealth shares of survivors.

For specific investment functions polynomial (5.3.21) can be simplified and explicit stability conditions can be derived. In particular, when all investment functions are horizontal in point $-\bar{e}$, the no-arbitrage equilibrium (if it exists) is always stable.

5.4 Market Selection and Asymptotic Dominance

The stability analysis performed in Section 5.3.3 provides an important information about the asymptotic conditions and the relative performances of the different traders interacting in the market. Inspired by the famous Friedman's hypothesis one could expect that if there is an agent who makes a better use of the information revealed by the trading activity and of the common and perfect knowledge about the fundamental process, this more "rational" trader will outperform the others and ultimately drive them out of the market. How and to what degree does this hypothetic result valid inside our framework? A different but related question is whether and to what extent the "dynamic" selection mechanism, which leads one type of agents to survive and possibly dominate the market, is beneficial to the economy as a whole, i.e. whether it generates a higher growth rate of the total wealth.

Let us consider a stable many agents equilibrium with price return r^*. According to the results of the stability analysis, the wealth return of all survivors is equal to r^*, so that r^* is also the asymptotic growth rate of the total wealth[4]. At the same time, the wealth growth rates of the non surviving agents are not higher than r^*. Then, if they were surviving, and consequently were affecting the dynamics of the total wealth, the whole economy could not grow with a higher rate. To put the same statement in negative terms, the economy will never end up in equilibrium where its growth rate is lower than what it would be if the survivor(s) were substituted by some other agent(s). One could see in this result an *optimal selection*

[4]This statement holds in all types of equilibria, irrespectively on whether $r^* = -\bar{e}$ or not. Particularly, in the former, no-arbitrage equilibrium, the economy and also the wealth of all agents are growing with zero (rescaled) growth rates.

principle since it suggests that the market endogenously selects the best aggregate outcome.

This result is in line with the intuitive idea that in a model with endogenous wealth dynamics, the agent who invests more in the growing asset increases his influence and, eventually, dominates those traders who invest less (Blume and Easley, 1992). Our analysis confirms in part this intuition, but also highlights two important limitations. First, the optimal selection principle does not apply to the whole set of equilibria, but only to the subset formed by the equilibria associated with stable fixed points in the single agent case. For instance, with the investment functions shown in Fig. 5.1, the market will never end up in U_1, even if this is the equilibrium with the highest possible return, since this equilibrium would not be stable if the survivor were present alone in the market. Second, the possibility to have *multiple stable equilibria*, even with one single trader, implies that the optimal selection principle has only a local character: the economy is not necessary attracted by the stable equilibrium with the highest possible return. Consider, for instance, the nonlinear investment function shown in the left panel of Fig. 5.3. This function possesses two stable equilibria: S_L and S_H. The ultimate equilibrium selected by the dynamics when this is the only function in the market will depend on the initial conditions; there are no guarantees that the market will end up in S_H, the equilibrium associated with the higher growth rate of the aggregate wealth.

The existence of multiple equilibria is also related to a second important implication of our analysis: the fact that the dominance of one investment function on another is a “local” property and depends on the market initial conditions. Consider a simple case with two heterogeneous traders. Suppose that their investment functions are the ones depicted in the right panel of Fig. 5.3 so that two stable equilibria exist: S_M and S_H. In the former equilibrium the agent with horizontal investment function dominates, while in the latter equilibrium he is dominated by another agent. Now assume that these two agents enter the market subsequently. It is immediate to see that it is *the order* in which these two agents enter the market that determines the final aggregate outcome and decides the ultimate survivor. This fact is not limited to some peculiar function, rather it is a general property of the model.

For any given investment function, it is always possible to build a second function which makes the equilibria of the first unstable and breaks its possible domination over the market. For instance, it would be misleading to conclude that the agent who invests the largest share of wealth in the risky security, possesses, due to the endogenous determination of prices, some “permanent” advantage in the speculative struggle. Consider a constant investment function with investment share $x = 1-\epsilon,\ 0 < \epsilon \ll 1$. Even if it possesses a stable equilibrium with very high return $r^* \sim 1/\epsilon$, it can be destabilized by a second investment function with constant investment share greater than one. Notice, however, that this second function does not become the winner: indeed, the equilibrium created by this function is unstable in the presence of the first. Rather, the presence of both functions in the market is likely to generate big fluctuations in returns and high volatility.

The impossibility to define the best or even a ”good” investment function independently of the set of investment functions present in the market reminds several results in evolutionary finance (see e.g. Hens and Schenk-Hoppé (2005b)) and is in tune with the discussion about the *limits to arbitrage* found inside the behavioral finance literature (see e.g. Barberis and Thaler (2003)). At the same time, it is in contradiction with the Friedman’s hypothesis: any ”rational” investment function, for instance derived from an utility maximization procedure, can be destabilized by some ”irrational” function, like the one characterized by a constant investment share.

Our results extend to an endogenous price setting the conclusions in DeLong, Shleifer, Summers, and Waldmann (1990b, 1991), obtained in simple models with exogenous price dynamics, that rational traders are unable to completely drive away from the market the noise investors. Intuition suggests that in an endogenous price formation setting, when the wealth of the agent feeds back into the model and affects the determination of future price returns, the survival probability of non optimal traders should be increased. We provide formal ground to this intuitive idea. Indeed, we prove that in the endogenous setting the relative importance of successful (irrational) trading is so high that not only the rational trader fail to dominate his irrational competitors, but also can be eventually dominated by them.

5.5 Conclusion

In this Section we analyzed the model in the case when an arbitrary number of heterogeneous agents with arbitrary technical CRRA-consistent behavior are operating in the market. We found all possible equilibria and showed how they can be characterized geometrically. We also performed the stability analysis of these equilibria.

We showed that our general results provide a simple and clear description of the principles governing the asymptotic market dynamics resulting from the competition of different trading strategies. We also demonstrated that the existence of multiple stable equilibria implies a local nature for the market selection process and, ultimately, the impossibility to define any global dominance order relation among agents. However, in the neighborhood of each equilibrium the market displays an "optimal" behavior leading to the dominance of those investment functions that guarantee to the whole economy a higher growth rate.

This market optimality does not have anything in common with optimality of the individuals, however. Indeed, since our framework is not restricted to those investment behaviors which are generated by expected utility maximization but can also include those which are favored by behavioral finance models, the results imply that rational (either in the sense of behavior or in the sense of expectations, or both) traders can be driven out from the market by irrational traders.

The present analysis can be extended in many directions. First of all, even if we proved that the existence of multiple equilibria is possible, the dynamics in this case remains to be unveiled. Probably numerical methods can be effectively applied to clarify the role of the initial conditions, the determinants of the relative size of the attraction domains for different equilibria, etc. These methods can also be used to study the dynamics in the cases when there are no stable equilibria.

Second, one may ask what are the consequences of the optimal selection principle for a market in which the set of strategies is not "frozen", but instead is evolving in time, plausibly following some adaptive process. For instance, one can assume that agents imitate the behavior of other traders (see e.g. Kirman (1991)) or that they update strategies according to recent relative performances (see e.g. Brock and Hommes (1998)). In such cases, those situations which we referred as "non-generic" above may become, instead, typical. Proposition 5.3.4 can be considered as a first step in the analysis of such situations.

Third, inside our general framework, numerous different specifications of the traders strategies are possible, in addition to the ones analyzed here. They range from the evaluation of the "fundamental" value of the asset, possibly obtained from a private source of information, to a strategic behavior that try to keep in consideration the reaction of other market participants to the revealed individual choices. The analysis of the consequences of the introduction of such strategies on the optimal selection principle may, ultimately, refute the statement about the impossibility of defining a dominance relation among strategies.

Relatedly, Assumption 2 about dividend yield process can be modified. We assumed that the yield is randomly and independently drawn from a stationary distribution. It would be interesting to consider the extension of the model for the case when the dividend follows an exogenous geometrical random walk and agents in their decisions are taking into account not only realized return but also fundamental price, analogous to the model in Chiarella, Dieci, and Gardini (2004).

Chapter 6

CRRA Endogenous Model with Linear Strategies

The purpose of this Chapter is to illustrate an application of both quantitative and qualitative results of general, analytically treatable agent-based model developed in three previous Chapters. In this way we continue the second example from Section 4.5, which demonstrated a possibility to study, inside our framework, the consequences of heterogeneity on the market stability.

For the application we choose the case of market with, roughly speaking, "linear" investment functions. Such behavior characterizes, for instance, those agents who derive their portfolio from the mean-variance optimization problem. In this respect, remember, that our general results suggest that there is no any link between rationality of the agent and his success. However, inside the literature on the heterogeneous agent modeling there is a tendency to perform the analysis in the framework with individual demand derived from utility maximization. Therefore, the analysis of this Chapter aims to clarify which features of the long-run market dynamics can be lost with such demand specification, and which, instead, are typical for the market with rational agents. Another goal of this Chapter is to show that the model developed in Chiarella and He (2001) can be completely reproduced in our general framework.

This Chapter is organized as follows. In the next Section we remind that the solution of an expected utility maximization problem with power utility leads to the investment functions which can be incorporated in our framework. Since this solution does not exist in the explicit form, we consider two possible approximations which appeared recently in the literature. We find that both these approximations provide investment functions which are linear under symmetrization. In order to understand what are the consequences of such feature we, in Section 6.2, apply the machinery developed in Chapter 5 to the case of single agent with linear investment function. In particular, we demonstrate that the phenomenon of multiple stable equilibria cannot emerge in such market. This is an important limitation of the "rational" framework with respect to the general case, especially if one believes that rationality should prevail in the market. In Section 6.3 we come back to the mean-variance type of behavior and consider those strategies which were introduced in Chiarella and He (2001). Through the re-consideration of the examples analyzed in that paper, we show that so-called "quasi-optimal selection principle" introduced there is a consequence of peculiar market ecology. For general behavior such principle does not hold and only local optimal selection principle formulated in Section 5.4 is valid. Some final remarks are given in Section 6.4.

6.1 "Rational" Investment Functions

In Section 3.2 we showed that the expected utility maximization problem (3.2.4) with power utility function (3.2.7) leads to the solution, which does not depend on the current wealth and price. Therefore, all such solutions (depending on the agent's perception about next period price return) can be represented by some investment functions satisfying Assumption 2″ from Section 5.1. Since the precise functional shapes of these investment functions are unknown, one should rely on approximation of such "rational" solutions.

Such direction was chosen in Chiarella and He (2001) inside the framework which is identical (to within the notation) to the one outlined in the previous Chapters. In particular, the market structure is the same and Assumption 2 holds. Model of Chiarella and He is written in terms of unscaled variables so that the gross (i.e. ex-dividend) return on the risky asset reads

$$r^G_{t+1} = \frac{P_{t+1} + D_{t+1} - P_t}{P_t} \quad . \tag{6.1.1}$$

Using the rescaled variables introduced in (3.5.7), this gross return can be expressed through the rescaled price return r_t as follows (cf. (3.5.8)):

$$r^G_{t+1} = r_f + (1 + r_f)\big(r_{t+1} + e_{t+1}\big) \quad , \tag{6.1.2}$$

which defines an isomorphic relation between our settings.

Chiarella and He consider the expected utility maximization with power utility and use the continuous time approximation. Under this approximation the individual investment share reads

$$x_t = \frac{1}{\gamma}\,\frac{\mathrm{E}_t\,[r^G_{t+1}] - r_f}{\mathrm{V}_t\,[r^G_{t+1}]} \quad , \tag{6.1.3}$$

where $\mathrm{E}_t[r^G_{t+1}]$ and $\mathrm{V}_t[r^G_{t+1}]$ stand for the agent's expectations about gross return (6.1.1) and its variance, respectively. Positive coefficient γ represents the relative risk aversion. Precise definition of investment share x_t depends on how the expectations are formed. The following specification is proposed

$$\mathrm{E}_t\,[r^G_{t+1}] = r_f + \delta + d\, m_t \tag{6.1.4}$$

$$\mathrm{V}_t\,[r^G_{t+1}] = \sigma^2\left(1 + b\,(1 - (1+v_t)^{-\xi})\right) \quad , \tag{6.1.5}$$

where m_t and v_t denote the sample estimates of the average return and its variance computed as equally weighted averages of the previous L observations

$$m_t = \frac{1}{L}\sum_{k=1}^{L} r^G_{t-k} \qquad \text{and} \qquad v_t = \frac{1}{L}\sum_{k=1}^{L}(r^G_{t-k} - m_t)^2 \quad . \tag{6.1.6}$$

Chiarella and He refer the reader to the contribution of Franke and Sethi (1998) for the justification of the choice (6.1.5) for the variance forecast. However, this choice and, in particular, parameters $b, \xi \geq 0$ are irrelevant for the equilibrium analysis, as we will see below. The specification of the expected conditional return (6.1.4) is important, on the contrary. It is defined as the risk free rate r_f plus the excess return. The latter is made of a constant component representing a risk premium, $\delta \geq 0$, and a variable component, $d\, m_t$. Parameter d is assumed to represent the way in which agents react to variations in the history of realized returns and can be used to distinguish between different classes of investors. A trader with $d = 0$ will ignore past realized returns and, consequently, can be thought as a *fundamentalist.* If $d > 0$ the agent can be considered a *trend follower*, if $d < 0$ he can be considered a *contrarian.* Example which we considered in Section 4.5.2 is inspired by this setting.

Estimates m_t and v_t depend on the previous L gross returns $r^G_{t-1}, \dots, r^G_{t-L}$. Then, after necessary substitutions of transformation (6.1.2), the expression in the right-hand side of (6.1.3) depends on the past L rescaled price returns. Therefore, this expression defines an investment function in the sense of Assumption 2″ of Chapter 5. We will refer to this function, which approximates the strategy based on the expected utility maximization, as f^{CH}. If we scale parameters δ and d dividing them on the minimal variance level σ^2, then this function reads:

$$f^{CH}(r_{t-1}, \dots, r_{t-L}) = \frac{1}{\gamma} \, \frac{\tilde{\delta} + \tilde{d}\, m_t}{1 + b\big(1 - (1 + v_t)^{-\xi}\big)} \,, \quad \text{with} \quad \tilde{\delta} = \frac{\delta}{\sigma^2} \,, \quad \tilde{d} = \frac{d}{\sigma^2} \,. \tag{6.1.7}$$

Functional form for the investment share x_t in (6.1.3) can, alternatively, be derived as a solution of the mean-variance optimization problem. This optimization will be analogous to (4.5.1). As we said above, Chiarella and He justify the function (6.1.3) on the basis of approximation of the solution coming from expected utility maximization[1]. It is interesting to mention that Campbell and Viceira (2002) derive another approximation for the solution of the same problem of expected utility maximization. This approximation differs from one in Chiarella and He (2001) on the constant term, and can be written as[2] $x'_t = x_t + 1/(2\gamma)$. Therefore, an alternative investment function $f^{CV} = f^{CH} + 1/(2\gamma)$ can be defined.

Both investment functions f^{CH} and f^{CV} represent different examples of the investment behavior compatible with our framework. These two examples are specific, however, because both investment functions become linear for the constant return history. Indeed, when the restriction $r = r_{t-1} = \dots = r_{t-L}$ is imposed for function f^{CH} of Chiarella and He, one gets $m_t = r^G$ and $v_t = 0$. Therefore, the symmetrization is linear with respect to r^G. Applying transformation (6.1.2), one gets linearity of the symmetrization also with respect to r. Then, the symmetrization of function f^{CV} of Campbell and Viceira is, of course, also linear.

Since, as we argued in the previous Chapter, the long-run outcome in the framework with CRRA trading behaviors crucially depends on the ecology of traders present in the market, the peculiar property of allowed investment functions casts some doubts on the results of the model in Chiarella and He (2001). It may be the case that the results which this model can generate are not general, but instead are consequences of specific market ecology. We will address this question in Section 6.3 where the model of Chiarella and He is discussed at length. Before that, it will be useful to take more general approach and analyze the market where investment strategies possess linear symmetrization but no other specifications are added. Such analysis will, first, prepare the ground for the re-investigation of Chiarella and He model, and, second, provide more extensive, comparing with example in Section 4.5.2, illustration of how our geometric machinery can be applied.

[1] See Appendix A.1 in Chiarella and He (2001). Notice that no estimation of the error incurred in considering the mean-variance approximation is provided.

[2] See formula (2.25) in Campbell and Viceira (2002).

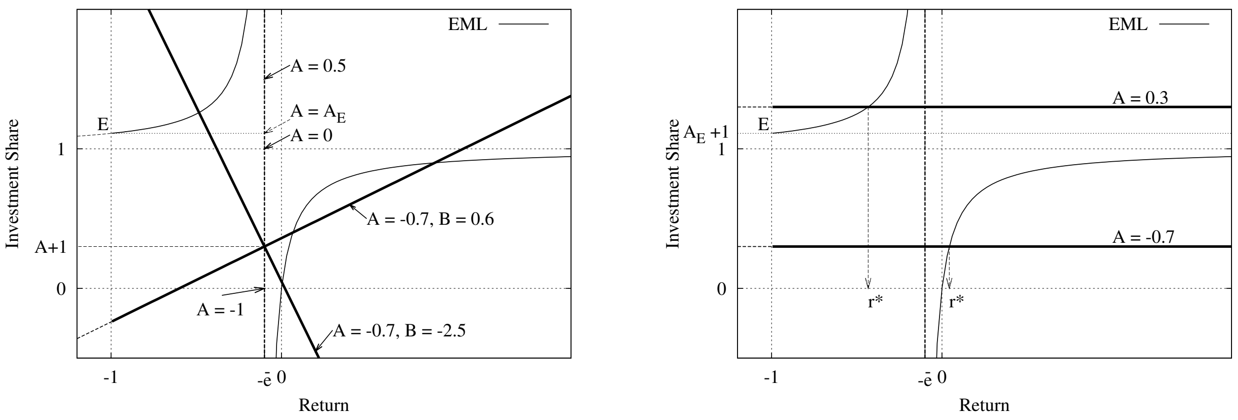

Figure 6.1: **Left Panel:** Examples of linear symmetrizations (6.2.1) for different parameters A and B. **Right Panel:** Equilibria for constant strategies.

6.2 Equilibria for Linear Investment Functions

Let us assume that investment function f is such that its restriction to the subspace defined as $r = r_t = r_{t-1} = \cdots = r_{t-L+1}$, is a *linear* function of r. It will be convenient to use the following parameterization of such restriction:

$$f(r, \ldots, r) = \big(A + 1\big) + B\,\big(r + \bar{e}\big) \quad . \tag{6.2.1}$$

Two parameters are involved in the description of this class of strategies. B stands for the *slope* of the symmetrization, while $A + 1$ gives the value of this function in point $-\bar{e}$. This parameterization is illustrated in the left panel of Fig. 6.1. Obviously, any strategy whose symmetrization is linear can be represented according to (6.2.1) with some A and B. In particular, the investment behavior considered in Chiarella and He (2001) can be described by (6.2.1) with coefficients defined as

$$A^{CH} = \frac{\tilde{\delta} + \tilde{d}\,r_f}{\gamma} - 1 \qquad \text{and} \qquad B^{CH} = \frac{\tilde{d}}{\gamma}\,(1 + r_f) \quad , \tag{6.2.2}$$

while those behaviors which are considered in Campbell and Viceira (2002) have $B^{CV} = B^{CH}$ and $A^{CV} = A^{CH} + 1/(2\gamma)$.

6.2.1 Location of Equilibria in Single Agent Case

Since both equilibrium and stability analysis of many agents case is based upon the corresponding analysis for the single agent, it will be enough to analyze the following system:

$$\begin{cases} x_{t+1} = f(r_t, r_{t-1}, \ldots, r_{t-L+1}) \\ r_{t+1} = \dfrac{x_{t+1} - x_t + \bar{e}\,x_t\,x_{t+1}}{x_t\,(1 - x_{t+1})} \end{cases} , \tag{6.2.3}$$

with investment function f characterized by (6.2.1). This system is a deterministic skeleton of system (5.2.2) whose equilibrium analysis was performed in Proposition 5.2.1.

We found, in particular, that all equilibria can be characterized as intersections of the investment function f and EML $l(r)$ introduced in Definition 5.2.1. From the geometric plot of the EML it is clear that depending on the values of A and B there exist at most two equilibria for the function possessing linear symmetrization. Simple computations confirm this inference. One has the following

Proposition 6.2.1. *Consider system* (6.2.3) *with single agent possessing the strategy with linear symmetrization* (6.2.1)*. Then the following cases are possible:*

(i) ***Constant strategy:*** $B = 0$.

For $A = 0$ there are no equilibria. If $A \neq 0$ there exist one equilibrium with return

$$r^* = -\frac{\bar{e}}{A} - \bar{e} \quad . \tag{6.2.4}$$

which is feasible, i.e. it generates positive price, when $A < 0$ and when $A > A_E = \frac{\bar{e}}{1-\bar{e}}$.

(ii) ***Non-Constant strategy:*** $B \neq 0$.

Consider $D = A^2 - 4B\bar{e}$. Then if $D < 0$, then there are no equilibria. Otherwise, when $D \geq 0$, there are two equilibria (coinciding when $D = 0$) with the following returns:

$$r_1^* = \frac{-A - \sqrt{A^2 - 4B\bar{e}}}{2B} - \bar{e} \quad , \quad r_2^* = \frac{-A + \sqrt{A^2 - 4B\bar{e}}}{2B} - \bar{e} \quad . \tag{6.2.5}$$

The equilibrium is feasible, i.e. it generates positive price, if the return exceeds -1.

Proof. See appendix F.1. □

This Proposition provides all possible equilibrium values of the return for different couples of parameters (A, B) for the strategies with linear symmetrizations (6.2.1). When $B = 0$ the agent's investment does not depend on the past information and his strategy represents the horizontal line as it is shown in the right panel of Fig. 6.1. If $A > 0$, in addition, the only equilibrium belongs to the upper-left branch of the EML, so that $r^* < -\bar{e}$. Obviously, this equilibrium is unfeasible when $A < A_E$. When $A < 0$ the equilibrium of constant strategy belongs to the lower-right branch of the EML, so that $r^* > -\bar{e}$ and it always generates positive prices.

When $B \neq 0$ one can distinguish between two cases. If the strategy is decreasing, so that $B < 0$, it is always the case that $D > 0$ and, therefore, two equilibria exist, as in example in the left panel of Fig. 6.1. From (6.2.5) it follows that $r_1^* > -\bar{e} > r_2^*$ in this case. Therefore, the first equilibrium belongs to the upper-left branch of the EML and might be unfeasible, while the second equilibrium is always feasible and belongs to the lower-right branch of the EML. In the opposite case, when $B > 0$, the strategy is increasing and one can have 0, 1 or 2 equilibria. In the latter situation, which is also illustrated in the left panel of Fig. 6.1, $r_1^* < r_2^*$ and both equilibria belong to the upper-left (lower-right) branch of the EML when $A > 0$ ($A < 0$).

In Fig. 6.2 we provide geometric illustration of all possibilities described in the last Proposition. Two upper panels represent examples of decreasing investment functions. In both cases two equilibria exist, either both feasible (the left panel, first row), or one feasible and one unfeasible (the right panel, first row). The second and third rows of panels provide four examples when both parameter A and slope B are positive. If two equilibria exist, then only the largest of them may be feasible (the left panel, second row). For higher values of B,

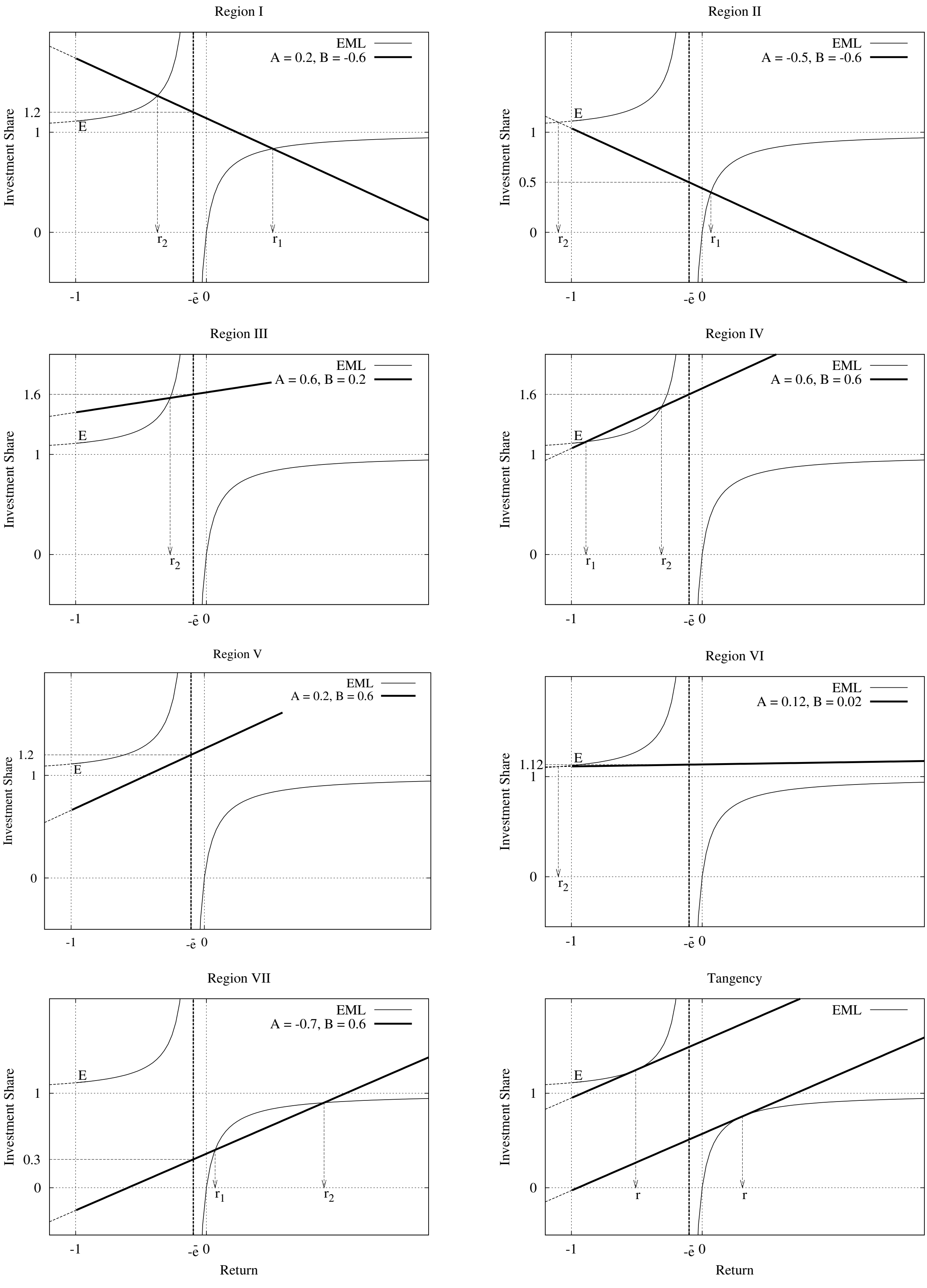

Figure 6.2: Examples of equilibria with linear strategies. The titles of the panels correspond to the regions in Fig. 6.3. See text for explanation.

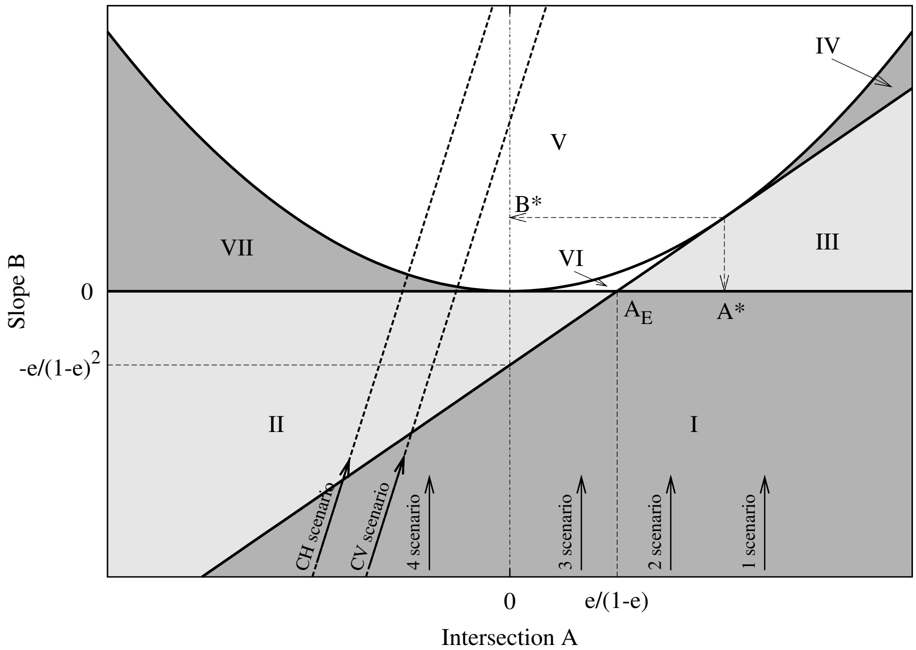

Figure 6.3: Stratification of the parameter space (A, B) according to the number of feasible equilibria. The dark gray (light gray, white) area represents the parameters for which there are two (one, zero) equilibria. See the text for the explanation of the regions marked by Roman numerals and Fig. 6.2 for the illustrative example for each of these regions. See the text for the explanation of the "scenarios" denoted by arrows in the down part of the picture and Fig. 6.4 for the illustrations.

i.e. steeper investment function, both equilibria become feasible (the right panel, second row). It is also possible that there exist no equilibrium (the left panel, third row), or that there exist no feasible equilibrium (the right panel, third row). Finally, two equilibria in the lower-right branch of the EML coexist when slope is positive but A is negative (the left panel, the last row). With increase of A or B, these two equilibria r_1^* and r_2^* approach each other and, eventually, coincide in the non-generic situations of tangency of the investment function and the EML (the right panel, the last row). With sufficiently high A both equilibria reappear through the tangency again. Such tangency scenario happens when $D = 0$ in Proposition 6.2.1*(ii)*.

Even more complete picture which illustrates Proposition 6.2.1 is presented in Fig. 6.3. The parameter space (A, B) is divided according to the number of different feasible equilibria. For the parameter pairs from the white area there are no feasible equilibria, for those pairs which belong to the light gray area only one feasible equilibrium exist, and, finally, if parameters belong to the dark gray area there exist two different feasible equilibria. Three important loci which are shown by the thick curves in Fig. 6.3 allow for more detailed stratification of the parameter space, so that seven different regions marked by the Roman numerals can be defined[3]. The first locus is a horizontal straight line corresponding to $B = 0$. In this case the investment strategy is horizontal and one equilibrium exist. Any change of B leads to the appearing the second equilibrium which can be unfeasible, though. The curve with parabolic shape contains the points with $A^2 = 4B\bar{e}$, i.e. those parameters for which the equilibrium is unique due to the tangency between the EML and the straight line (6.2.1). This parabola, therefore, separates the parameters for which there are no equilibria (region V) from those

[3]Each of the first seven panels in Fig. 6.2 gives the example for the corresponding region in Fig. 6.3.

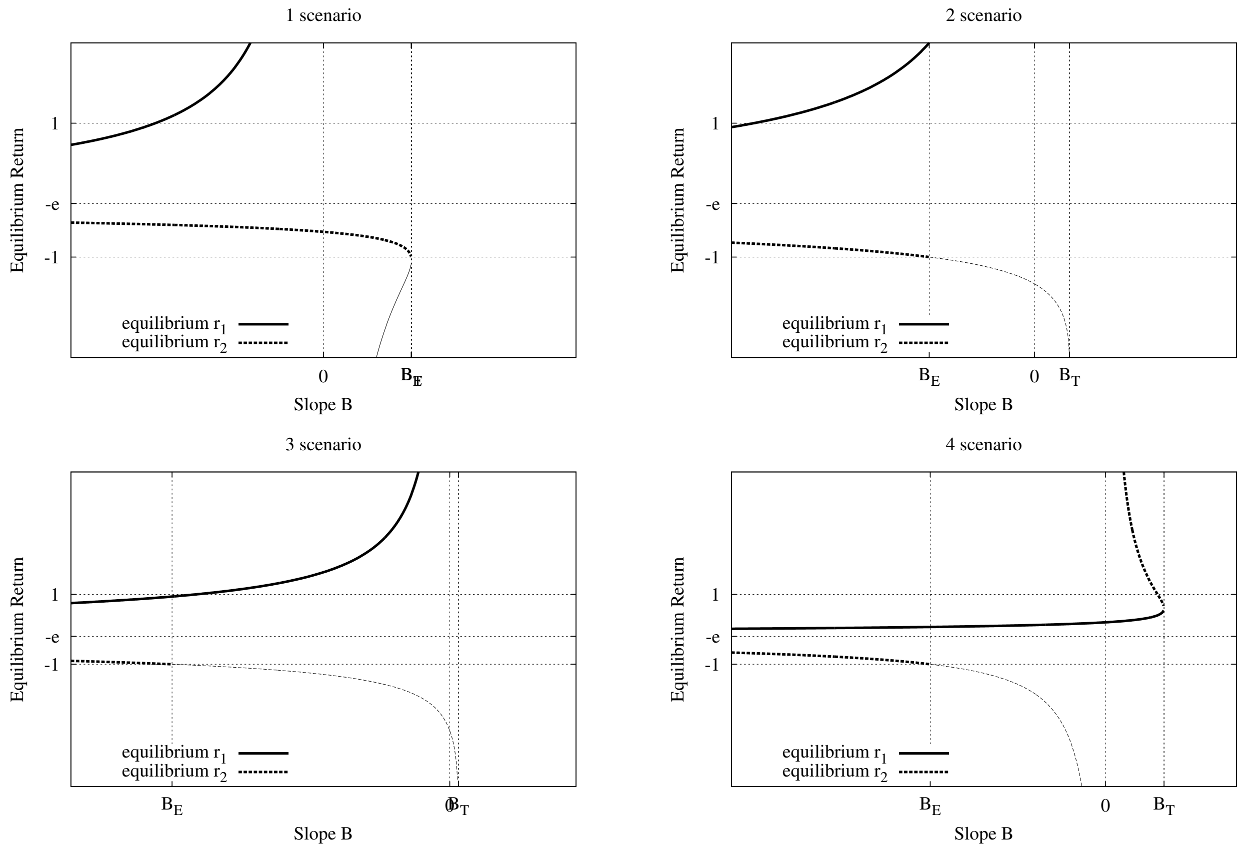

Figure 6.4: Equilibria r_1^* and r_2^* as functions of the strategy's slope B for four different levels of A. **Upper Left Panel**: $A > A^*$. **Upper Right Panel**: $A^* > A > A_E$. **Lower Left Panel**: $A_E > A > 0$. **Lower Right Panel**: $A < 0$.

points for which two equilibria exist. We call this line "existence line".

Finally, the third locus is represented by an increasing straight line $A = \bar{e}/(1-\bar{e}) - B(\bar{e}-1)$, corresponding to the parameter pairs for which linear symmetrization (6.2.1) passes through the end-point E of the upper-left branch of the EML. With the crossing of this locus, which we call "feasibility line", one feasible equilibrium is lost. If B is negative, the equilibrium on the upper-left branch of the EML disappears with decrease of A, so that two regions I and II are determined. If $B > 0$ and $A < 0$ then, as we mentioned above, both equilibria (if exist) belong to the lower-right branch of the EML and both are feasible, so that area VII is determined. Finally, if both A and B are positive, let us denote as (A^*, B^*) the parameters defining the strategy which passes through E and, at the same time, tangent to the EML. In region III only the equilibrium with the smallest return r_1^* is feasible. When A decreases there are two possible options: either r_1^* also becomes infeasible or r_2^* becomes feasible. From the EML plot it is easy to see that the first case happens for $B < B^*$, i.e. when in the end-point E the strategy is flatter than the tangency line. In this case from region III we move to region VI. Respectively, the second case happens when $B > B^*$ and we move from region III to region IV.

The EML can be effectively used to study the effects of change of different parameters. For instance, if the value of A is fixed, then increase of B from $-\infty$ to $+\infty$ corresponds to the counter clock wise rotation of the vertical line $-\bar{e}$ around the point with the ordinate

$A+1$ on the graph in Fig. 6.1. One can easily sketch the graphs of the roots behavior like we do in Fig. 6.4. These four graphs can, alternatively, be understood from Fig. 6.3. We fix abscissa A and move up vertically through this figure (see the arrows in the down part of the picture). Let us denote the intersections with the feasibility and existence lines as B_E and B_T, respectively. One can identify the following four possible scenarios, therefore.

If $A > A^*$ the sequence of different regions one goes through with the increase of B reads $I \to III \to IV \to V$. This situation is depicted in the upper-left panel of Fig. 6.4. When $A_E < A < A^*$, the sequence becomes $I \to III \to VI \to V$ as in the upper-right panel of Fig. 6.4. When $0 < A < A_E$, then we have the sequence $I \to II \to VI \to V$ and corresponding equilibria plot is shown in the lower-left panel of Fig. 6.4. Finally, when $A < 0$ one crosses $I \to II \to VII \to V$ and gets the graph on the lower-right panel of Fig. 6.4. In all cases both equilibria approach $-\bar{e}$ with $B \to -\infty$, which is obvious from the EML plot.

6.2.2 Stability of Equilibria for $L = 1$ Case

We address in this Section the question of equilibrium stability for the single agent case. Unfortunately, the general conditions provided by Proposition 5.2.2 cannot be simplified with the use of the only assumption that the investment strategy possesses a linear symmetrization. One issue here is that such assumption does not provide any information about L partial derivatives of function f in equilibrium, which appear in the stability conditions through the characteristic polynomial P_f defined in (5.2.5). Even if this polynomial is simplified somehow, there is another issue to make explicit the requirement for its roots to be inside the unit circle.

The stability conditions can be obtained in explicit form for the case $L = 1$ (see Proposition 4.3.2). Therefore, we strengthen here the assumption about linearization (6.2.1) exploited in the previous analysis and assume that the investment function of the agent reads:

$$f(r) = \big(A+1\big) + B\,\big(r+\bar{e}\big) \quad . \tag{6.2.6}$$

Linear investment choice based on a naïve forecast of the future return represents one possible interpretation of such behavior.

The stratification in Fig. 6.3 showed the number of different equilibria, in general, and feasible equilibria, in particular. The question about their stability leads to the following conditions derived from Proposition 4.3.2:

$$\frac{B - l'(r^*)\,r^*}{r^*} < 0\,, \qquad B - l'(r^*) < 0 \qquad \text{and} \qquad \frac{B(2+r^*) + l'(r^*)r^*}{r^*} > 0\,, \tag{6.2.7}$$

where r^* stands for the equilibrium return. Corresponding values of the return were computed in Proposition 6.2.1. In the case when strategy is constant, the unique equilibrium has return r^* provided by (6.2.4). When the strategy is not constant, two equilibria returns are given by (6.2.5). Plugging the values of returns in (6.2.7), one can express stability conditions and bifurcation locuses through parameters A and B. The resulting expressions are quite cumbersome, so we provide only their geometric illustration.

In Fig. 6.5 we consider the parametric space (A, B) and produce its stratification in accordance to the validity of the stability conditions. More precisely, in each point in the space we compute all possible feasible equilibria and check whether each of three inequalities (6.2.7) holds in these equilibria. First four panels correspond to the stability of the root r_1^* defined in (6.2.5), while the other four panels illustrate the stability of r_2^*. In the first and fifth panels we explore the validity of the first inequality in (6.2.7) for the corresponding root. In the gray

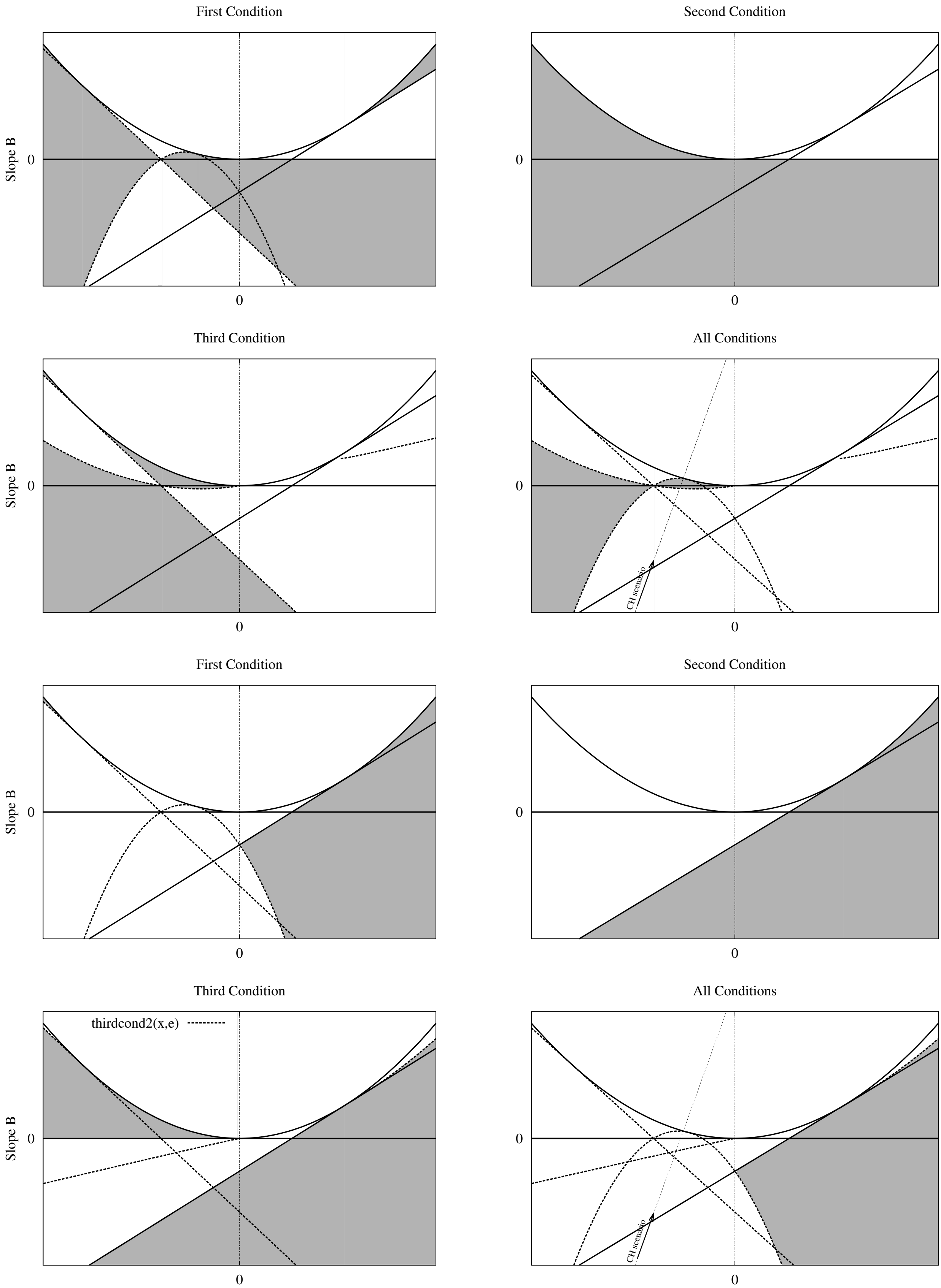

Figure 6.5: Stratification of the parameter space (A, B) according to the stability of equilibria. **First four panels**: stability of the first root r_1. **Last four panels**: stability of the second root r_2. Corresponding root is stable if parameters belong to the gray area in the fourth and eighth panels. See titles of the panels and text for explanation.

regions the root exist, it is feasible and first inequality is satisfied. Otherwise, the parameter couple belongs to the white region. We repeat this procedure for the second inequality (second and sixth panels) and the third inequality (third and seventh panels). Finally, the fourth and eighth panels answers the question about the stability of the corresponding equilibrium. The gray regions there contain only those points where all three inequalities (6.2.7) are satisfied.

The different regions in all these pictures are separated by thick lines. Apart from two loci representing "existence" and "feasibility" lines which separated the regions in Fig. 6.3, one can recognize *bifurcation loci* (shown as dotted thick lines) corresponding to point where different inequalities in (6.2.7) turn out to be equalities. For example, the straight line and convex parabola, appearing in the first and fifth panels, correspond to those points where the first inequality changes its sign. Therefore, in these points the system exhibits the Neimark-Sacker bifurcation. Analogously, the concave parabola in the third and seventh panels represents points of flip bifurcations.

Fig. 6.5 allows to easily grasp the effects of different parameters on the stability of equilibria. One can, for example, repeat the same procedure as we applied to Fig. 6.3 when we studied four different scenarios, emerging when intersection A is fixed and slope B is changing. For instance, it is immediate to see that in the first three scenarios, where $A > 0$ and which were represented by the first three panels in Fig. 6.4, equilibrium r_1^* is unstable for any value of B, while equilibrium r_2^* is stable when the absolute value of B is small enough. Furthermore, in the latter case, if B increases then the equilibrium exhibits flip bifurcation, while when B decreases there is Neimark-Sacker bifurcation. In the fourth scenario with negative A, equilibrium r_2^* is unstable, while r_1^* is stable for B close to zero.

Apart from geometric application, Fig. 6.5 suggests that even if two feasible equilibria can coexist for linear investment functions, at least one of them is unstable. For the case of increasing linear investment functions it is, indeed, obvious from the EML plot. If such strategy intersects the EML twice as in Fig. 6.2 for regions IV and VII, then in one of these intersections the slope of the strategy B is greater than the slope of the EML, and, therefore, the second inequality in (6.2.7) holds with the opposite sign. Generally we have the following

Proposition 6.2.2. *There is at most one feasible stable equilibrium in the market with single linear investment function* (6.2.6)*.*

Proof. See appendix F.2. □

This Proposition is the main result of this Section. It makes clear that the restriction of the analysis in the markets populated by the agents with linear strategies (in particular, those who derive their demand through mean-variance optimization) leads to the impossibility to have the phenomenon of multiple *stable* equilibria in the single agent case. If non-linear strategies were allowed, many stable equilibria could co-exist as we discussed in Section 5.4. We reproduce corresponding geometric example in the left panel of Fig 6.6.

As a consequence of this limitation, the range of possible market dynamics can be over-simplified if only "linear" behaviors are considered. This is clear in the single agent case. Furthermore, it can also be the case in the market with many agents, since the equilibria and their stability in such market are characterized through the equilibria and stability of the single agent equilibria. For instance, one can be easily convinced[4] that the market with many

[4]Consider the lower-right branch of the Equilibrium Market Line and remember that the equilibrium will be unstable in any of two following cases. First, if there exist more aggressive strategy in this equilibrium. Second, if the increasing strategy intersects the EML from below.

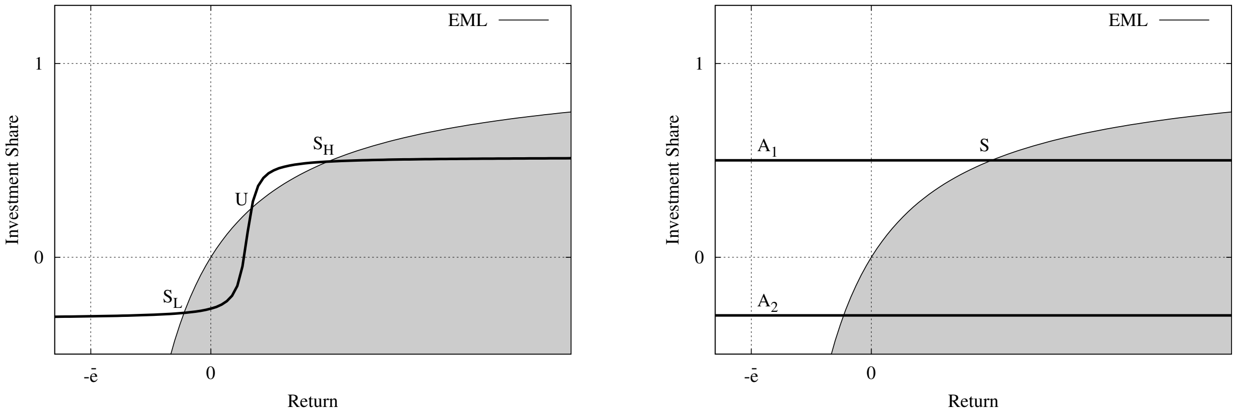

Figure 6.6: **Left panel:** In the market with single agent multiple stable equilibria can be generated by non-linear investment function. S_H and S_L are stable while U is unstable. **Right panel:** In the market with two agents multiple stable equilibria can be generated with linear strategies. S is stable with one survivor. (A_1, A_2) generates unique stable equilibrium where both agents survive.

agents having linear strategies cannot possess more than one stable equilibrium with $r^* > -\bar{e}$. Another example will be discussed in the next Section.

It is important to stress that Proposition 6.2.2 does not, in general, hold in the market with many linear strategies. It can be seen from the situation depicted in the right panel of Fig 6.6, where the market possesses one stable equilibrium S with one surviving agent and also another no-arbitrage equilibrium where two agents survive (see Propositions 5.3.2 and 5.3.5). Both equilibria are asymptotically stable, since strategies are horizontal.

6.3 CRRA Framework with Mean-Variance Approximation

Geometric interpretation of equilibria and investigation of particular case with linear investment strategies developed in the previous Section can now be straight-forwardly applied to the analysis of the market with "rational" strategies derived from mean-variance optimization and introduced in Section 6.1. Such analysis has been performed in Chiarella and He (2001). First of all, let us review the results of this model.

6.3.1 Model of Chiarella and He: Review of Results

Chiarella and He consider agents with investment function f^{CH} given in (6.1.7). All these agents have the same risk aversion coefficient $\gamma = 1$, i.e. the same demand function. Two different cases are analyzed. The first case is the model with homogeneous expectations. In this model the realized demand functions of all agents are identical. They are characterized by (rescaled) risk premium $\tilde{\delta} \in (0, 1)$. Accordingly with the sign of the extrapolation parameter $\tilde{d}$ the situations of fundamental, trend-following or contrarian behavior described in Section 6.1 are possible. Chiarella and He provide complete equilibrium analysis in each of these cases (Proposition 3.1). Stability analysis is performed for the case $\tilde{d} = 0$, when the unique equilibrium is asymptotically stable and for the case when $\tilde{d} \neq 0$ and $L = 1$ when sufficient conditions for stability are derived (Corollary 3.3). For larger L the numerical approach is exploited which shows that the stability can be brought to the system through increase of the "memory span". The qualitative aspects of the equilibrium and stability analysis are summarized in Figure 1 of that paper.

After the analysis of the homogeneous expectations case, Chiarella and He proceed to the market with two different investors. They consider four possible scenario. In the first scenario there are two fundamentalists with different risk premia in the market. In formal terms it means that $\tilde{d}_1 = \tilde{d}_2$, but $\tilde{\delta}_1 \neq \tilde{\delta}_2$. The equilibrium analysis of this case shows that there are two equilibria in such market (Proposition 4.2), however only one of them is stable (Corollary 4.3). It leads Chiarella and He to formulate "optimal selection principle" for such scenario. It states that the investor with the higher risk premium will be the winner (Figure 2).

The second scenario corresponds to the market with one fundamentalist and one contrarian. There exist three steady-states for such market, but (unscaled) price return is positive in only two of them (Proposition 4.4). The fundamentalist dominates the market in one of these two steady-states and contrarian dominates the market in another one. The stability analysis can be performed analytically only for the former steady-state (Corollary 4.5), so that Chiarella and He analyze the stability of the latter steady-state numerically. As a result they conclude that the long-run return dynamics depends on the relative levels of the returns in these two steady-states and follows a similar optimal selection principle. Namely, the steady-state is stable if it generates the highest return (Figure 3).

In the third example of heterogeneous market fundamentalist meets trend-follower. Such market has the equilibrium where fundamentalist survives and may also have zero, one or two equilibria with surviving trend-follower (Proposition 4.6). Similar to the previous example, the stability conditions for the latter equilibria are obtained through numerical investigation. It is found that for small extrapolation rates (i.e. for relatively small value of $\tilde{d}$ of the trend-follower) there exist two equilibria where trend-follower survives. The highest return is generated in one of these equilibria which is, however, unstable. Between the two remaining equilibria "the stability switching follows a (quasi-)optimal selection principle", depending where the return is higher (Figure 4).

Finally, in their last example Chiarella and He consider the market with two chartists. In this case there exist multiple steady states. If traders extrapolate strongly (i.e. in particular they both are trend-followers) none of the steady-states is stable. For weak extrapolators, "the stability of the system follows the (quasi-)optimal selection principle – the steady-state having relatively higher return tends to dominate the market in the long run" (Figure 5).

To summarize, Chiarella and He have found *quasi-optimal selection principle* which allows to predict a long-run market dynamics in the case, when there are multiple equilibria. Comparing this principle with optimal selection principle which was formulated in our Section 5.4, one can identify an important difference.

Indeed, the principle of Chiarella and He has a *global* character. When the ecology of the traders is fixed, it can be applied to the market, so that unique possible outcome is predicted. Our optimal selection principle has a *local* character, instead. For a given traders' ecology there can be different possibilities of the market long-run behavior, i.e. multiple equilibria. The final outcome depends on the initial conditions and, in the stochastic case, on the yield dynamics, and cannot be predicted *a priori*. However, independently of the realized equilibria, the survivors will be chosen in "optimal" way: to allow the highest possible growth rate of the economy in this point. In some sense, our principle selects among strategies, while principle of Chiarella and He chooses among equilibria.

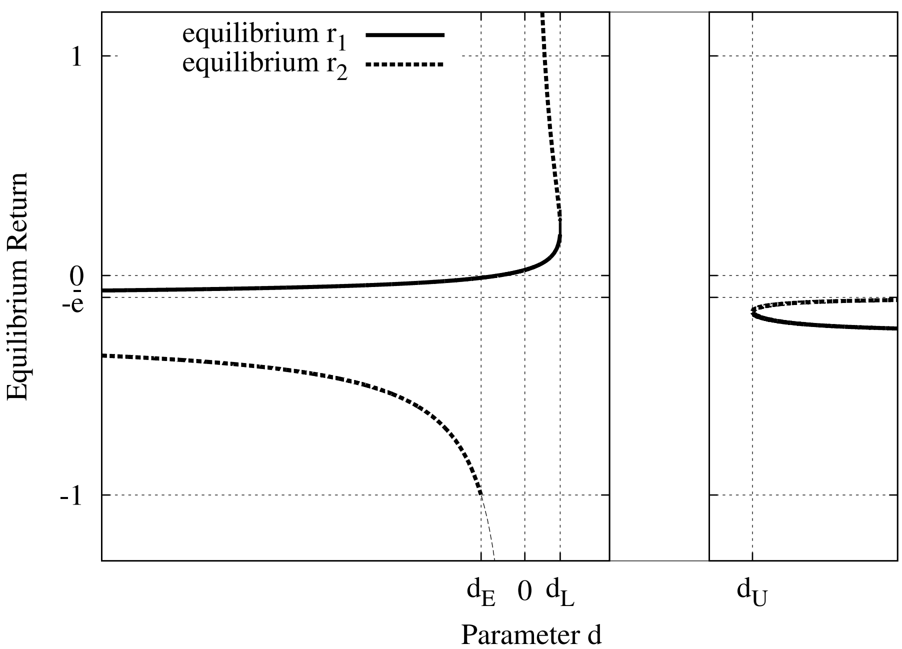

Figure 6.7: Equilibria in the model of Chiarella and He as function of extrapolation parameter d.

The discussion in the previous Section suggests that the ultimate reason for the global character of the "quasi-optimal selection principle" lies in the specific demand functions analyzed in Chiarella and He (2001). In particular, the model of Chiarella and He cannot have multiple stable equilibria, which is not general feature of such framework.

6.3.2 Model of Chiarella and He: Geometric Approach

Let us show that analytical and a bit cumbersome results in Chiarella and He (2001) may become much more clear if one uses the geometric tools. In (6.2.2) we have computed the coefficients which, in terms of parameterization (6.2.1), define strategies allowed in the model of Chiarella and He. Now we can use these relations to study the impact of different parameters on equilibria and their stability.

We start with the single agent situation. On the stratification diagram of Fig. 6.3 all strategies in Chiarella and He with fixed risk aversion coefficient γ, risk premium $\tilde{\delta}$ and risk-free interest rate r_f can be represented by the dotted straight line with positive slope which we label as "CH scenario". The bottom-up movement along this line corresponds to increase in $\tilde{d}$. This parameter reaches the zero value in the point $B = 0$. The results of equilibrium analysis of Proposition 3.1 in the paper of Chiarella and He can now be reproduced straight-forwardly.

Indeed, the dotted line subsequently intersects regions I, II, VII, V and IV in Fig. 6.3. When the rate of extrapolation of the contrarian is high (in absolute value), parameters belong to the region I and, therefore, two feasible equilibria coexist. With increase of extrapolation parameter, the "feasibility line" is intersected for some $\tilde{d} = d_E$. At this point one of two equilibria becomes unfeasible. The remaining feasible equilibrium is unique for all $\tilde{d} \in (d_E, 0]$. When $\tilde{d} > 0$, the agent is trend-follower and parameters belong to region VII. Here again there are two coexisting equilibria. With further increase of the rate of extrapolation, the "existence line" is intersected for some $\tilde{d} = d_L$ and both equilibria disappear. In region V there exist no

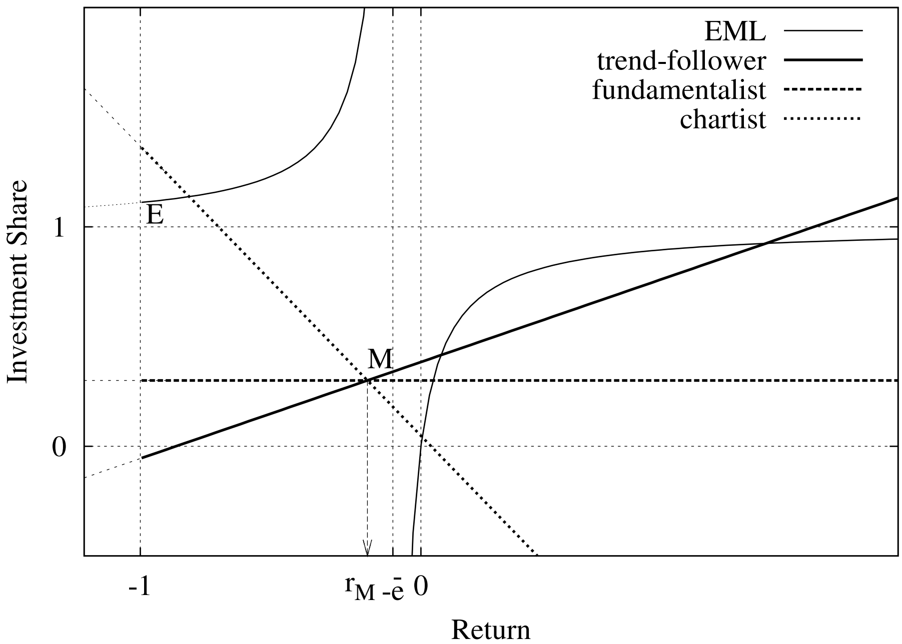

Figure 6.8: Equilibria on the EML for contrarian, fundamentalist and trend-follower strategies considered in Chiarella and He (2001).

equilibria, but when extrapolated parameter is very high, i.e. agent extrapolates strongly, the "existence line" is intersected second time in some point $\tilde{d} = d_U$. After this intersection two equilibria coexist again. As a result of such consideration, we reproduce (and improve) Figure 1 in Chiarella and He (2001) in Fig. 6.7.

Alternative, and in a sense, more explicit way to understand the graph in Fig. 6.7 is to exploit the Equilibrium Market Line. In order to do it we notice that symmetrization (6.2.1) of function f^{CH} always passes through the point[5]

$$M = \left(r_M,\ \tilde{\delta}\,\right) \quad , \quad \text{where} \qquad r_M = -\bar{e} - \frac{r_f}{1+r_f} \quad , \tag{6.3.1}$$

which does not depend on $\tilde{d}$. The slope of the symmetrization is equal to $\tilde{d}\,(1+r_f)$. Three typical behavior are presented in Fig. 6.8. The horizontal investment function corresponds to $\tilde{d} = 0$, i.e. to the fundamentalist type of behavior in terms of Chiarella and He. Analogously, any trend-follower possesses an increasing strategy, while the chartist's investment function is decreasing. Notice also that return r_M in (6.3.1) corresponds to the zero level of gross unscaled return.

Counter clock wise rotation of the straight vertical line passing through point M immediately explains Fig. 6.7. In particular, notice that d_E represents the value of extrapolation parameter, when the corresponding investment function of contrarian passes the end point E of the upper-left branch of the EML. Also values of d_L and d_U correspond to the trend-followers whose strategies are tangent to the EML.

Stability analysis of equilibria which Chiarella and He performed for the case $L = 1$ can be easily illustrated in Fig. 6.5. In particular, any horizontal (fundamental) strategy is stable,

[5]We consider case $\gamma = 1$, but the generalization for the other cases is trivial.

and such equilibrium remains to be stable for $\tilde{d}$ close to 0. Moreover, equilibrium r_1^* is stable for very small negative $\tilde{d}$, while equilibrium r_2^* is stable for very large positive $\tilde{d}$.

Further advantages of the geometrical application of the EML can be seen if one considers market with N agents. As we described in Section 6.3.1, Chiarella and He confine the analysis to the case $N = 2$ and consider four different scenario: 2 fundamentalists, fundamentalists vs. contrarian, fundamentalists vs. trend-follower, and two chartists. We illustrate all these possibilities in Fig. 6.9. In such geometric way, all the results of the model Chiarella and He (2001) can be easily re-obtained.

Consider, first, the case of two fundamentalists with different risk premia $\tilde{\delta}_1 > \tilde{\delta}_2$ (the left panel, first row). These traders have horizontal strategies passing through points M_1 and M_2 defined in (6.3.1). From the assumption on the risk premia it follows that M_1 is higher than M_2. There are two equilibria in such market: S and U. Each of these equilibria would be stable if the corresponding agent would operate alone. When two agents operate together, then equilibrium S with the highest risk premium is stable, while U is unstable.

Let us now suppose that fundamentalist with risk premium $\tilde{\delta}_1$ encounters on the market contrarian with risk premium $\tilde{\delta}_2$. In other words, horizontal and decreasing strategies are competing. Chiarella and He distinguish between two cases depending on which of these risk premia is higher. Geometrically, it corresponds to the location of points M_1 and M_2.

We start with the case in which $\tilde{\delta}_1 \geq \tilde{\delta}_2$, i.e. when point M_1 is higher than point M_2 (the right panel, first row). With respect to the previous case we have made a rotation of the lower strategy around point M_2. It is obvious that equilibrium S_f is always stable in this case, while equilibrium S_c cannot be stable. First graph in Figure 3 of Chiarella and He paper illustrates qualitative features of this situation. Notice, however, that the return in equilibrium S_c *does not* approach the return in equilibrium S_f when $\tilde{d}_2 \to 0$, on the contrary to what is drawn on that Figure.

If $\tilde{\delta}_1 < \tilde{\delta}_2$, there are different possibilities. If contrarian extrapolates not very strongly, so that an absolute value of $\tilde{\delta}_2$ is small enough (left panel, second row), then S_f is, certainly, unstable equilibrium. Therefore S_c remains to be the only candidate for the stable equilibrium on two-agents market. It will be stable only when it is stable in the single agent case, which happens for relatively small $\tilde{d}_2$ (see fourth panel in Fig. 6.5). Otherwise, there is no stable equilibria in the market. If, on the other hand, contrarian extrapolates strongly (the right panel, second row), then S_f is the only stable equilibrium. Comparing this analysis with the second graph in Figure 3 in Chiarella and He (2001), we can see that the situation of possible absence of any stable equilibrium in the market have been overlooked in that paper.

In the third example we consider the case when horizontal strategy of fundamentalist with risk premium $\tilde{\delta}_1$ competes against an increasing linear strategy of the trend-follower with risk premium $\tilde{\delta}_2$. In this example, Chiarella and He again distinguish between two cases depending on which of the risk premia is greater.

Let us, first, assume that $\tilde{\delta}_1 \geq \tilde{\delta}_2$. There are two possibilities. If the trend follower extrapolates not too strong, equilibrium S_t is not stable (the left panel, third row). The equilibrium S_f is stable in this case. If the trend follower extrapolates stronger, his investment function rotates and equilibrium S_f looses its stability. S_t remains to be the only candidate for the stable equilibrium. If it exist and stable in the market with trend-follower alone, it is also stable in the two-agents situations (the right panel, third row). Otherwise, there are no stable equilibria in the market with two agents. It definitely the case for $d_U > \tilde{d}_2 > d_L$, since in this situation there is no equilibrium in the market with surviving trend-follower. But it

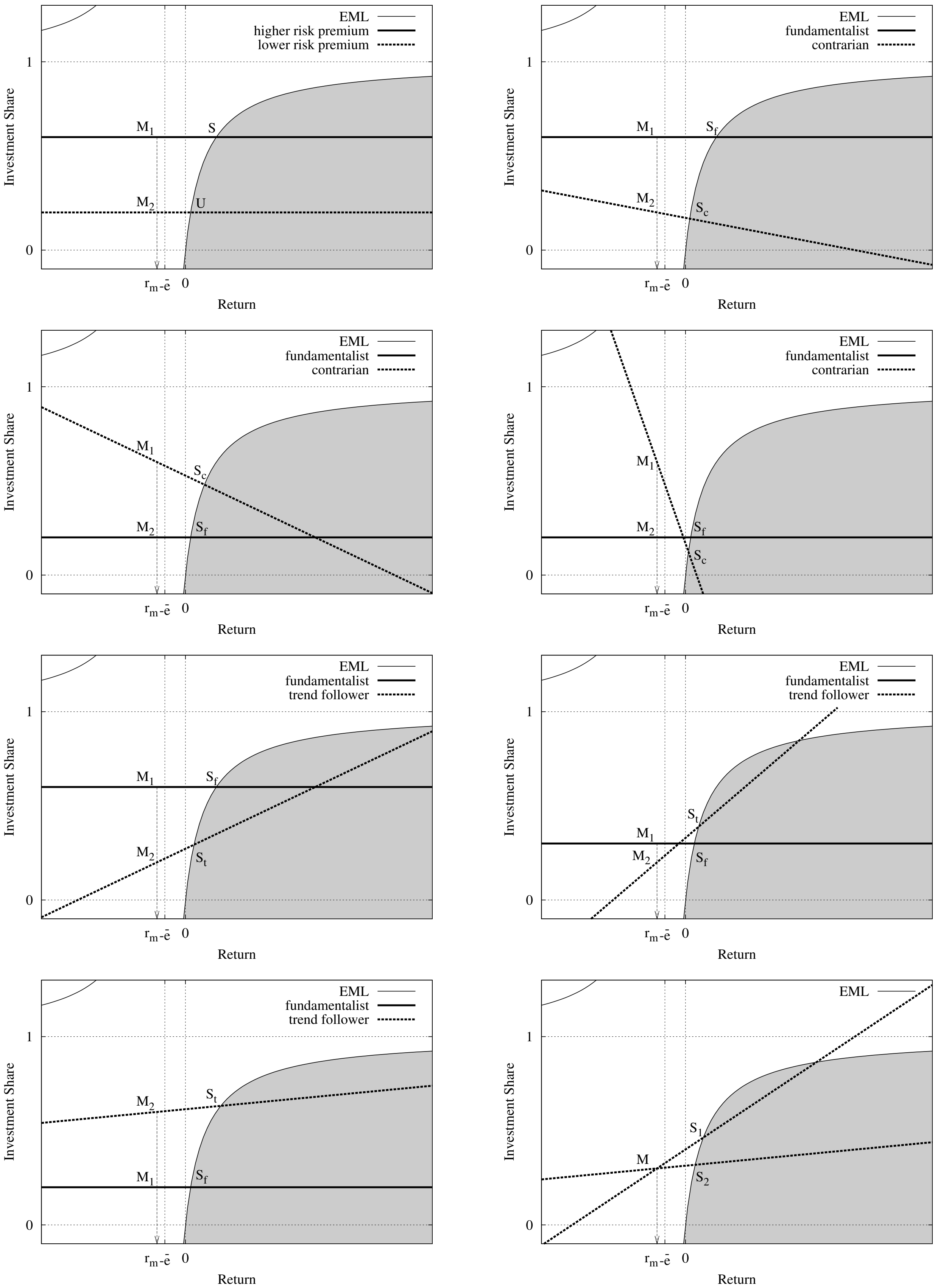

Figure 6.9: Equilibria in the model of Chiarella and He with two agents. See text for the explanation.

also happens for some $\tilde{d}_2$ lower than d_L. Finally, for very strong extrapolation, when $\tilde{d}_2 > d_U$ the market may have a stable equilibrium, if it exists for trend-follower.

In the case when $\tilde{\delta}_1 < \tilde{\delta}_2$ (the left panel, fourth row) it is obvious that equilibrium S_f cannot be stable, therefore market will have a stable equilibrium S_t whenever it is stable for trend-follower, that is for small enough $\tilde{d}_2$. On the base of this discussion one can immediately recognize that the graph in Figure 4 in Chiarella and He (2001) is not correct.

Finally, in the right panel in the fourth row of Fig. 6.9 we consider an example when two technical traders coexist in the market. We draw the situation when both of them are trend-followers and have the same risk premium, so that their strategies pass through the same point M. It is clear, that the agent with the lowest extrapolation rate will generate equilibrium S_2 which will always be unstable. Instead, equilibrium S_1 generated by the second agent will be stable if and only if it is stable in the single agent market. Compare it with panel (b) in Figure 5 in Chiarella and He (2001) and notice that with further increase d_2 the stable equilibrium (with growing return) becomes unstable and, eventually, disappears. So that for higher extrapolation rates market does not have any equilibrium.

6.4 Conclusion

In this Chapter we have shown that heterogeneous agent model presented in Chiarella and He (2001) can be easily reproduced by means of the analysis developed in the previous Chapters. It is clear that we generalized this model in many directions. The number of agents is not two, but arbitrary. The demand of the agents is not peculiar but arbitrary. And the expectations are not necessary based on the equally weighted forecast but can be arbitrary. Furthermore, the stability conditions are derived analytically for arbitrary long "memory" in the case of exponentially weighted averages (Proposition 4.3.6), and corrected optimal selection principle is proven analytically in the most general case (Proposition 5.3.4).

However, the easiest and, probably, the most impressive way to illustrate the advantages of our general approach is to have a look at Fig. 6.3 and 6.5. Remember that these stratification diagrams are drawn for a very particular case of agent's behavior, for the case in which the market cannot display multiple equilibria. Even in this particular case, the scope of the model of Chiarella and He is represented by the one-dimensional straight line marked as "CH scenario". Furthermore, only small interval in this line is analyzed in that model, since risk premium $\tilde{\delta}$ is assumed to be bounded inside an interval $(0, 1)$. This is the reason why Chiarella and He do not have no-arbitrage equilibria in their model.

Chapter 7

Stochastic Dynamics and Large Market Limit

In this Chapter we provide a simulation analysis of the stochastic versions of our system. Remember that previous investigation concerned the deterministic skeleton of the corresponding dynamics. We obtained deterministic dynamical system, substituting the random dividend yield by its mean value, and analyzed the equilibria and their stability for this system. Our intuition was that the stochastic dynamics can be approximated by such deterministic skeleton. If the latter generates the dynamics, for instance, converging to the fixed point, then the former should generate noisy dynamics which is, in the long run, concentrates around this fixed point. And *vice versa*, if the skeleton predicts divergent dynamics, then the initial stochastic system should also diverge from the equilibrium. The question which immediately arises and which we pose in the first part of this Chapter is whether this intuition is valid, i.e. whether the deterministic skeleton does approximate the stochastic dynamics.

Analysis of this Chapter is also concerned the economic issue of the impact of noise in individual micro-behavior on the aggregate macro-dynamics. The question, whether the micro noise is important for the macro level or, instead, it is washed out as in a sort of "central limit theorem", always attracted the highest attention in economics, both from the theoretical and empirical point of view (see e.g. Forni and Lippi (1998)). In terms of our model one can ask what if the agents are "almost identical" in their behavior, that is their investment functions are the noisy versions of some average choice. Will the notional, "representative" agent, characterized by such average choice, correctly represent the multi agent dynamics if the number of agents is sufficiently large? Or, instead, agent specific noise will propagate and the system will be far from the trajectory of this average agent? To put the same question in another terms, we are interested in an identification of the precise "boundaries" outside which the *representative agent* approach fails. This question is important in the light of the extensive critics to the use of representative agent in the economic models, see e.g. Kirman (1992) or Gallegati and Kirman (1999).

Apart from economic interest, answers on these questions will substantially increase our setting. We are now in the possessing of complete knowledge about the equilibria of the system in the case when there is an arbitrary number of agents with different and "frozen" behaviors. The analysis of this Chapter will extend our knowledge to the case of the market with arbitrary number of different *clusters* of the agents, where inside each cluster there is an infinite number of similar agents' behaviors.

The rest of this Chapter is organized as follows. In the next Section we present the analysis of the impact of the stochastic dividend on the dynamics. We compare the simulations for the stochastic system and deterministic skeleton and infer that deterministic approximation works. This result provides a ground for all our previous analysis. In Section 7.2 we formalize the ideas outlined above about noisy micro-behavior. We approach this issue from a broader perspective and, first, identify general conditions which are sufficient for the characterization of equilibrium return dynamics. Then, we notice that these conditions are satisfied in the setting where the aggregate outcome of stochastic individual choices can be described as a deterministic function of the state of the market. Finally, we provide some examples when such situation can be observed and develop an intuition why it should be the case when the investment choices of different agents are independent random deviations from a common behavior and the number of agents operating on the market tends to infinity. We call such scenario "Large Market Limit". In Section 7.3 we analyze the validity of the "Large Market Limit" with the help of numerical analysis. Section 7.4 concludes.

7.1 Dynamics with Stochastic Yield

In order to explore the dynamic effect of the random dividends we performed a bunch of simulations, both for system (5.3.3) and for its deterministic skeleton, with different market ecologies. Some of the simulations are presented in Fig. 7.1. The general result is that simulations confirm the intuition and suggest that the deterministic skeleton gives a reasonable approximation of the dynamics. Stability of equilibrium is not disturbed if the noise is small enough and the smaller this noise is, the better the approximation. Moreover, in the case of multiple stable equilibria the long-run dynamics depends not only on the initial conditions, but also on the stochastic yield in the transitory path.

In all simulations which we discuss below, random dividend yield $\{e_t\}$ is modeled as uniformly distributed on the interval $(\bar{e} - s_e, \bar{e} + s_e)$. The mean of the dividend yield is fixed as $\bar{e} = 0.04$, while the support s_e is changing in order to analyze the role of the yield variability on the dynamics.

We start with the single agent case and consider the same linear investment function which have been used in our illustration of the system dynamics in Chapter 4. There, in the left panel of Fig. 4.3, we have shown that investment function $f(r) = 0.199 + 0.1\,r$ generates the fluctuations converging to the equilibrium with (rescaled) return $r^* = 0.01$. Let us compare the dynamics of the deterministic skeleton of the stochastic system with the dynamics generated by this system, see the left panel in the first row in Fig. 7.1. The thick line reports the trajectory of the skeleton, while dotted lines represent the trajectories of the stochastic system for two different yield ranges s_e. We can see that both during the transitory path and after the equilibrium had reached, the skeleton does approximate the stochastic dynamics of the system. Moreover, as expected, the less the yield variability, the better this approximation. Notice also that the size of the noise around equilibrium is as expected. In the skeleton,

$$r^* = \bar{e}\,\frac{x^*}{1 - x^*} = 0.25\,\bar{e} \quad ,$$

so that return of stochastic system, presumably, should belong to $(0.01 - 0.25\,s_e, 0.01 + 0.25\,s_e)$. This is confirmed from simulations.

In the second example, which we present here, there are two agents with constant investment functions $f_1 = 0.5$ and $f_2 = -0.25$. In such situation there are two stable equilibria of the deterministic skeleton (see the right panel in Fig. 6.6). In the first equilibrium the first agent survives and his behavior determines equilibrium return $r_1^* = \bar{e} = 0.04$, while in the

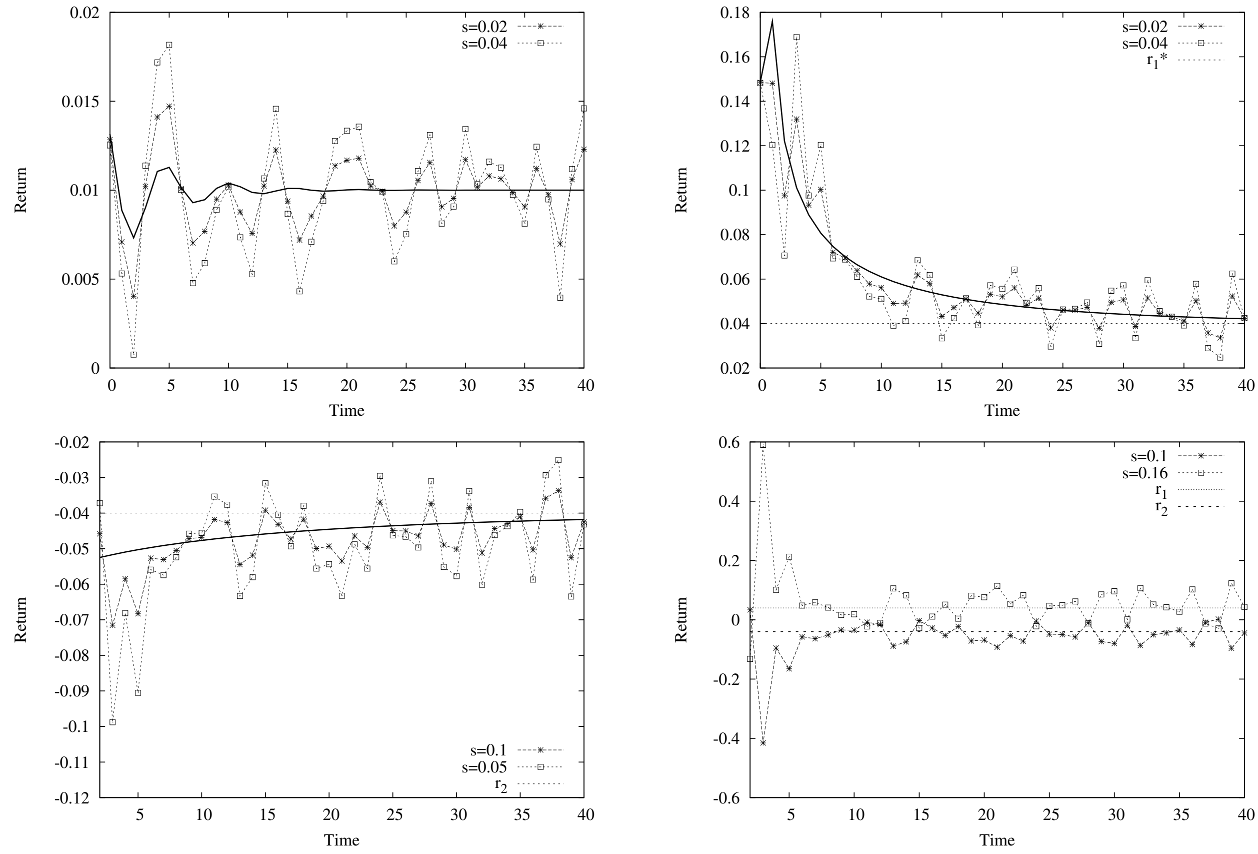

Figure 7.1: Trajectories of stochastic system with one or two agents. The dividend yield adds noise in the return time series but does not change the long-run behavior of the system with respect to the dynamics of the deterministic skeleton. See text for explanation.

second, no-arbitrage equilibrium both agents survive and return $r_2^* = -\bar{e} = -0.04$. In the right panel in the first row of Fig. 7.1 we demonstrate the case in which the initial conditions are such that the skeleton converges to r_1^* (shown as dotted horizontal line). The stochastic system again generates the noisy dynamics which, after transitory stage, is concentrated around the equilibrium. Another simulation for the same system is presented in the left panel in the second row of Fig. 7.1. The initial price is now such that deterministic system converges to no-arbitrage equilibrium r_2^*. The same happens also for the stochastic system.

The situation depicted in the last graph is, however, more complicated. It turns out that observed agreement between the deterministic skeleton's dynamics and stochastic trajectory is affected by the choice of the noise dynamics. As we show in the right panel in the second row of Fig. 7.1 (where the initial conditions are the same as in the previous example), if we choose another level of the yield variability, the dividend realizations during transitory period may lead to considerable change in the market dynamics. Indeed, when the noise is small and $s_e = 0.1$, both deterministic and stochastic system end up in the situation of the co-existence of traders. However, when $s_e = 0.16$ the stochastic system is attracted by another stable equilibrium and only one agent eventually survives. It is important to mention that the effect of the size of noise is not linear. Indeed, it is the realization of the yield process and not the level of its variance determines the long run outcome.

7.2 Large Market Limit

The agent-specific noise, i.e. stochastic term which is added to the agents' investment share, represents another type of randomness whose analysis on the system dynamics we would like to explore. In this Section we address this question from a broader perspective. Let us come back to the system derived in Proposition 3.6.1. According to (3.6.4), in the deterministic skeleton (rescaled) return r_t evolves as follows

$$r_{t+1} = \frac{\left\langle x_{t+1} - x_t \right\rangle_t + \bar{e} \left\langle x_t \, x_{t+1} \right\rangle_t}{\left\langle x_t \left(1 - x_{t+1}\right) \right\rangle_t} \quad . \tag{7.2.1}$$

The problem, which we analyzed in previous Chapters, was to find some fixed point of the return evolution, i.e. conditions under which the return evolution is invariant under time shift operator.

Remember that in accordance with (3.6.2), $\left\langle \cdot \right\rangle_t$ represents the weighted average of the argument, where weights are provided by the relative wealth shares of the agents, $\varphi_{t,n}$. By and large, return evolution depends on the evolution of the distribution of weights in (7.2.1) across population. This is why the problem of determination of a rest point of the dynamics (7.2.1) is not trivial.

Analysis in the previous Chapters illustrated one possible way to deal with such problem. We supplemented the return dynamics by an explicit evolution of the weights and figured out the dynamics of the corresponding multi-dimensional system. Since investment choices of the agents were assumed to depend on the return history, in equilibrium these choices were automatically invariant. This was not enough, however. For correct definition of equilibrium, the *distribution of weights* had to be invariant under time shift operator as well. It essentially explains why in all generic equilibria only one of the agents survived and why in all other equilibria, where two or more agents survived, the survivors had to have the same growth rates of their wealth.

It suggests that, in general terms, for the invariance of the return provided by (7.2.1), we need a fulfillment of two requirements. First is invariance of the investment shares computed on the distribution of weights

$$\left\langle x_{t+1} \right\rangle_t = \left\langle x_t \right\rangle_t \quad . \tag{7.2.2}$$

And, second, since the distribution is also evolving, we need an invariance of this distribution

$$\left\langle x_{t+1} \right\rangle_t = \left\langle x_{t+1} \right\rangle_{t+1} \quad . \tag{7.2.3}$$

The following statement, which links the past and present value of the weighted investment shares, allows to see the implications of this requirement about the distribution invariance.

Lemma 7.2.1. *The wealth-weighted average of market investment choices at time $t+1$, $\left\langle x_{t+1} \right\rangle_{t+1}$ can be computed using present wealth shares $\varphi_{t,n}$ according to*

$$\left\langle x_{t+1} \right\rangle_{t+1} = \left\langle x_{t+1} \right\rangle_t + \left(r_{t+1} + e_{t+1}\right) \Big(\left\langle x_{t+1} \, x_t \right\rangle_t - \left\langle x_{t+1} \right\rangle_t \left\langle x_t \right\rangle_t \Big) \quad . \tag{7.2.4}$$

Proof. See appendix G.1. □

Now, it is clear that if conditions (7.2.2) and (7.2.3) are satisfied for all t, then the return dynamics (7.2.1) can be simplified to

$$r_{t+1} = e_{t+1} \frac{\langle x_{t+1} \rangle_t}{1 - \langle x_{t+1} \rangle_t} \quad ,$$

which is, indeed, invariant under these two conditions.

In the course of this discussion we identified two conditions under which the appropriate equilibrium exists for the return dynamics. These conditions are not about individual agents as in the propositions in Chapters 4 and 5. They are expressed through some aggregate quantities and, therefore, more general. In particular, (7.2.2) and (7.2.3) are satisfied in those equilibria which we found and analyzed in previous Chapters, i.e. when there are arbitrary many agents in the market with appropriately defined investment shares. These condition can also be satisfied in some other situations. One of such situations is provided by the following

Assumption 2‴. There exist a deterministic function F such that

$$\langle x_t \rangle_t = F(r_{t-1}, r_{t-2}, \dots) \quad . \tag{7.2.5}$$

Moreover, the investment choices at successive time steps satisfy

$$\langle x_t \, x_{t+1} \rangle_t = \langle x_t \rangle_t \langle x_{t+1} \rangle_t \quad . \tag{7.2.6}$$

In this Assumption, which is as usually supplemented by Assumption 1 about yield distribution, we require that the average investment choice is described by some time-invariant function of past information. In addition, we assume that the sample average of the product of the investment shares can be replaced by the product of their sample averages. From Lemma 7.2.1 is follows that the last requirement is equivalent to condition (7.2.3), while equality (7.2.2) is automatically satisfied when return is in equilibrium.

When Assumption 2‴ holds the dynamics of the economy can be described in terms of the sole aggregate variable $\langle x_t \rangle_t$. It reads

$$\begin{cases} \langle x_{t+1} \rangle_{t+1} = F(r_t, r_{t-1}, \dots) \\ \qquad r_{t+1} = \dfrac{\langle x_{t+1} \rangle_{t+1} - \langle x_t \rangle_t + e_t \langle x_t \rangle_t \langle x_{t+1} \rangle_{t+1}}{\langle x_t \rangle_t - \langle x_t \rangle_t \langle x_{t+1} \rangle_{t+1}} \end{cases} \quad . \tag{7.2.7}$$

This system coincides with the dynamics generated by a single trader, and function F describes the "behavior" of this notional "representative" agent. Properties of this system for different specifications of function F were analyzed in Section 4.3 for the case when past information can be mapped into the future choice through the EWMA estimators and in Section 5.2 for the general case.

The natural question which remains to be answered is under which circumstances Assumption 2‴ can be considered a reliable description of a multi-agent system? Obviously, it is satisfied in the homogeneous case, when all agents possess the same beliefs and preferences, so that at each time step $x_{t,n} = \langle x_t \rangle_t$, $\forall n$. This example does not enlarge our knowledge about system, however.

A more interesting example is constituted by the case of "purely noisy" agents. Suppose that at each time step the investment shares of the N agents are randomly and independently drawn from a common distribution with average value $\bar{x}$. In this case the information set is irrelevant and in the $N \to \infty$ limit, if the share of wealth of each agent goes to zero, one has $\langle x_t \rangle_t = \bar{x}$. In this case, (7.2.6) is clearly fulfilled, so that conditions of Assumption $2'''$ are replicated, and the dynamics of returns reduces to

$$r_{t+1} = e_{t+1} \frac{\bar{x}}{1 - \bar{x}} \quad .$$

As expected, in the deterministic skeleton, return r^* and average share $\bar{x}$ are linked through the EML relation $\bar{x} = l(r^*)$. When N is finite, the dynamics is noisy even in the deterministic skeleton. However, all such random dynamics will, presumable, concentrate around r^*, at least when N is sufficiently large. Concerning the role of the noisy yield, we showed in Section 7.1 that it does not affect the average dynamics provided by the skeleton in the limiting case. Then, it should not be too important for large N as well.

We can generalize the "pure noise" model and assume that the investment decision of each agent is a "noisy" version of a basic common choice; formally

$$x_{t,n} = F(r_{t-1}, r_{t-2}, \dots) + \epsilon_{t,n} \quad , \tag{7.2.8}$$

where the ϵ's are independent (across time and across different agents) random variables with zero mean. When $N \to \infty$ the sample of independent random variables becomes large, the sample average converges to the average of the variable distribution, so that one can expect[1] that $\langle x_t \rangle_t \to F(\mathcal{I}_{t-1})$. If shocks are independent across time, one can also expect the fulfillment of (7.2.6) for the corresponding limits. Therefore, the dynamics is described by the system (7.2.7). Since any "theoretical" relation for the averages of random variables is violated on finite samples, we consider this behavior as a sort of "limiting" situation and call it a *Large Market Limit (LML).*

Return dynamics in the LML is approximated by the dynamics in the case of single agent present in the market. This is the reason why the existence of the LML can be linked with the representative agent approach. In general case, the representative agent in this sense, i.e. agent possessing some average behavior, does not represent the long-run economic outcome. For instance, if there are many agents in the market with constant choices inside interval $(0, 1)$, then the long-run dynamics is described as if only the agent with the highest investment choice would survive. Even if under this scenario, market can be described by means of one agent, this agent is not a "representative" agent in the sense of standard economic theory.

If, instead, the agents behaviors are described by (7.2.8), then the notional agent possessing investment function F is, indeed, representative in economic sense. The question remains whether this dynamics is close to the real one. We do not give a general answer to this non-trivial question, leaving it for future research. Instead, with the help of computer simulations, we show in the next Section that in the neighborhood of the equilibrium the LML can be, indeed, considered a satisfactory approximation of the actual multi-agents dynamics.

7.3 Numerical Analysis of the Large Market Limit

The setting of the model in the LML is characterized by two different sources of randomness: first, the idiosyncratic noise components $\epsilon_{t,n}$ appearing in the definition (7.2.8) of the agent investment choice and, second, the exogenous stochastic dividend process $\{e_t\}$. Each of them

[1] Note that all averages which we consider are weighted with respect to variables changing over time. Therefore we cannot straightforwardly use any limit theorem to justify our reasoning.

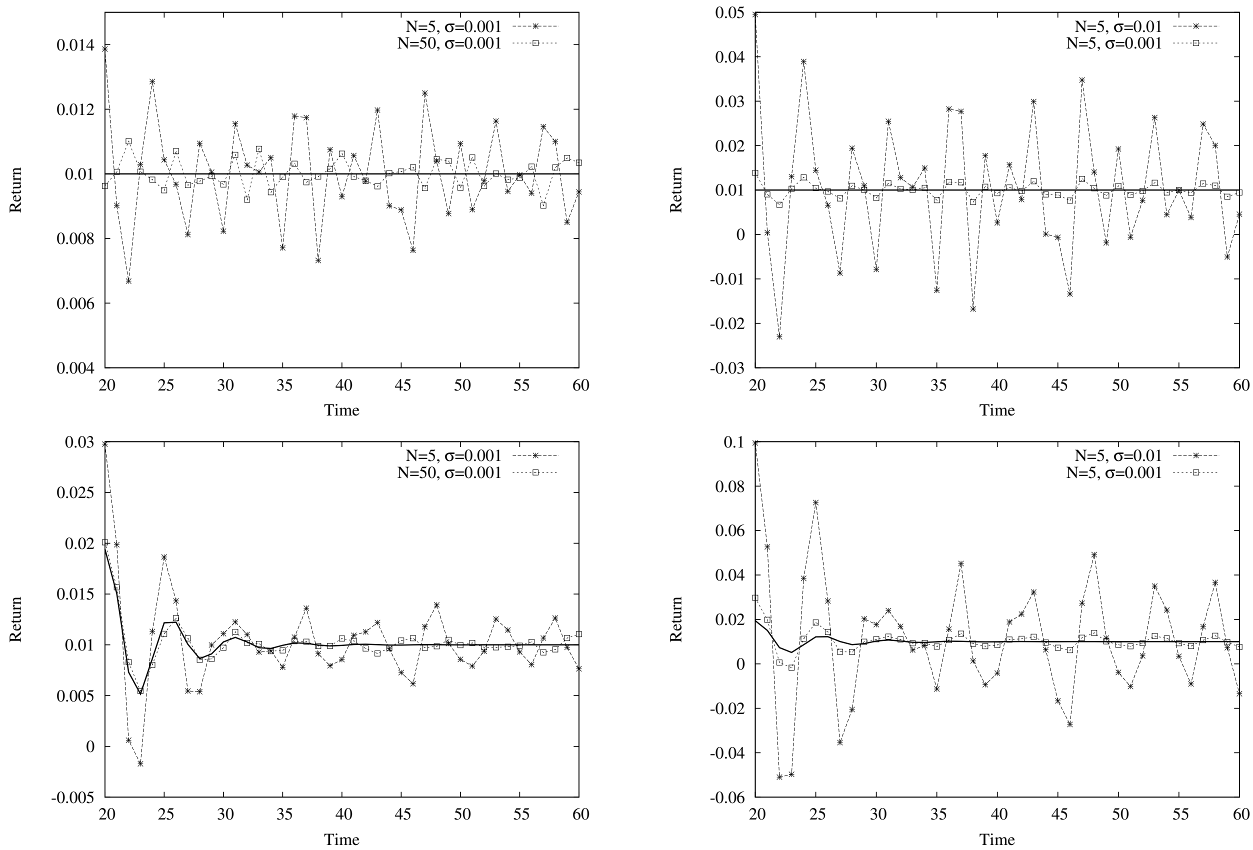

Figure 7.2: Convergence to the deterministic LML with increase of the number of agents (**Left Panels**) and with decrease of the variance of the agent-specific noise (**Right Panels**). Simulations are for pure noise model (**Upper Panels**) and for model with linear increasing strategy (**Lower Panels**).

may affect the dynamics. With the help of computer simulations we try to analyze the impact of both these noise sources here. We do in in two steps. In the first step we consider constant yield scenario and compare the dynamics generated by a finite population of noisy agents with the deterministic dynamics of the "representative" agent in the LML. In the second step we will do the same comparison in the stochastic environment, i.e. when dividend yield is random.

As we illustrate below, it turns out that even if differences between the limiting deterministic behavior and the actual stochastic implementation of the model do exist, they become smaller when the size of the population of agents increases, when the variance of the agent-specific noise $\epsilon_{t,n}$ decreases, and when the variance of the dividend process e_t decreases.

7.3.1 LML in the Deterministic Skeleton

Let us start with the case when dividend yield is constant and focus on the effect of the noisy components in agents investment function. We model noise $\epsilon_{t,n}$ in (7.2.8) as normally distributed independent random variable with zero average and standard deviation σ_ϵ. We confine the analysis on the case of linear functions. Notice that the same analysis, therefore, can be applied to the function of any type in the appropriately small neighborhood of the equilibrium.

The results of simulations are presented in Fig. 7.2, where the thick line represents the

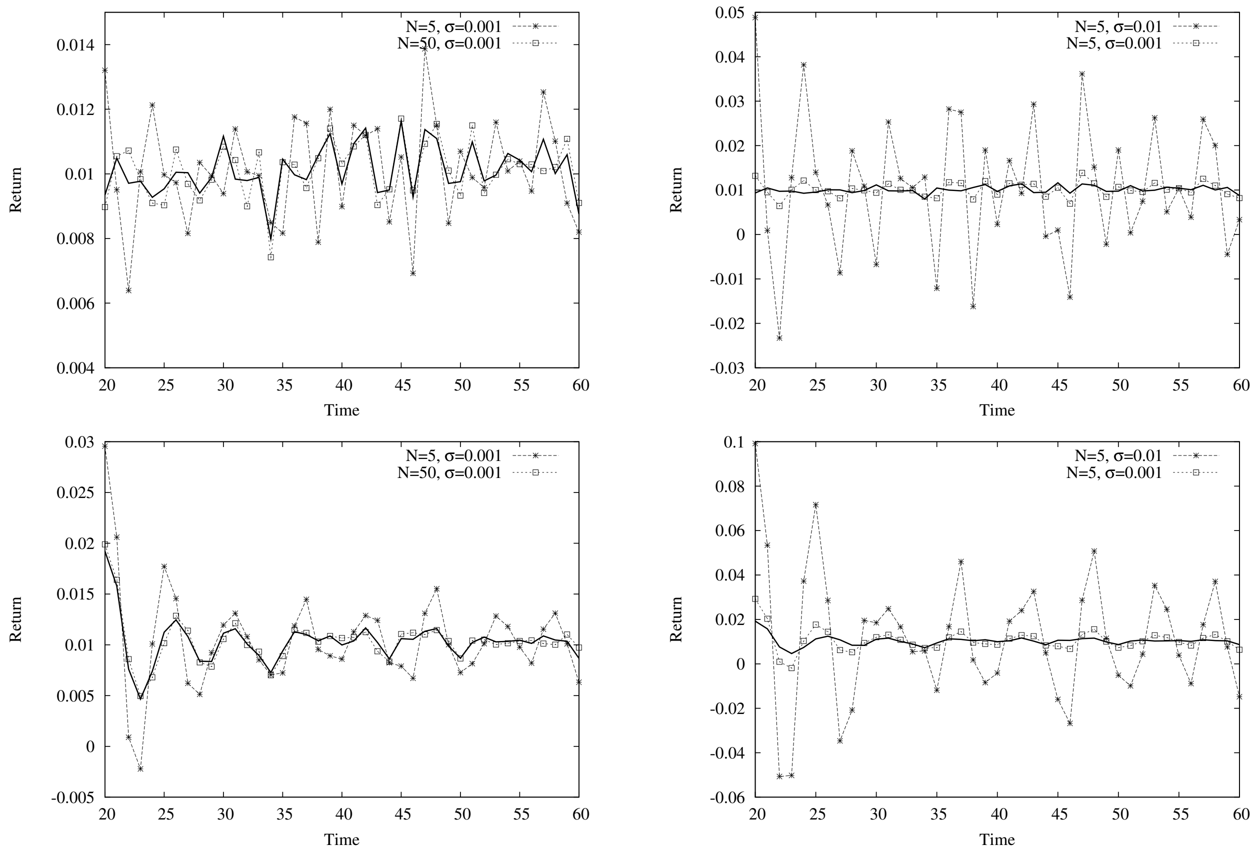

Figure 7.3: Convergence to the stochastic LML with increase of the number of agents (**Left Panels**) and with decrease of the variance of the agent-specific noise (**Right Panels**). Simulations are for pure noise model (**Upper Panels**) and for model with linear increasing strategy (**Lower Panels**).

dynamics in the LML. Since dividend yield is constant the LML gives a deterministic system in this case. We consider two examples, the pure noise model, which corresponds to (7.2.8) with constant function F (see two upper panels), and model with increasing linear function F (see two lower panels). For definiteness, in the first case we fix $F = 0.2$, while in the second case $F(r) = 0.199 + 0.1\,r$. In both cases the LML deterministic system possesses unique stable equilibrium with $r^* = 0.01$. When function F is constant, then convergence happens almost immediately, while if this function is increasing, there is some transitional period during 40 time periods or so.

Fig. 7.2 shows that if noise is added to the individual demand functions, the observed returns fluctuates, but these fluctuations are around the corresponding LML-trajectory and, in particular, around equilibrium after transitional period. In the left panels we illustrate the role of the population size, so the standard deviation of the noise terms σ_ϵ is fixed. We can that the variance of the fluctuations decreases with the increase of the population size N. In the right panels we, instead, fix the number of agents and consider two cases with different standard deviation σ_ϵ. If this deviation decreases, the variance of fluctuations decreases as well.

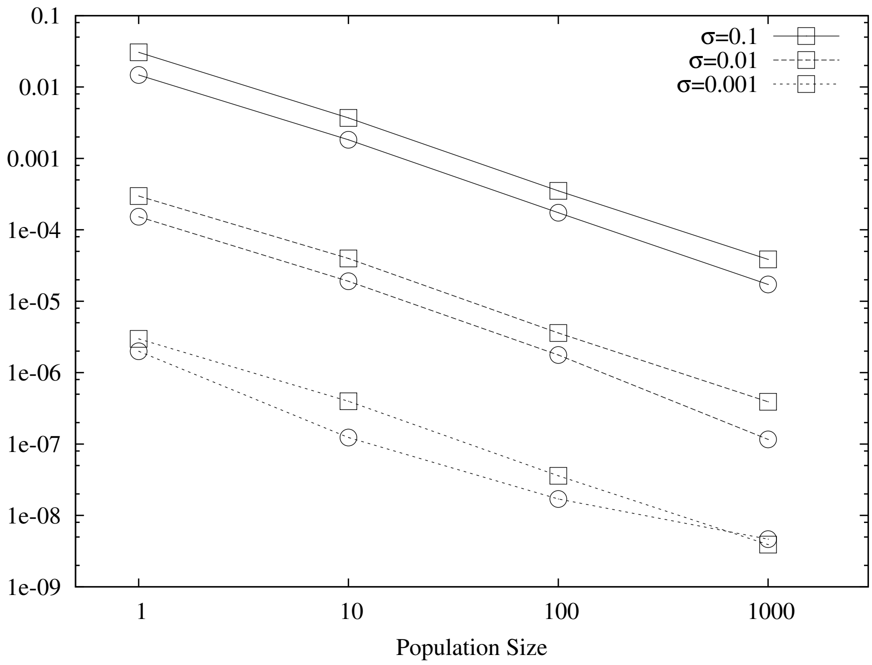

Figure 7.4: Average over 100 simulations of the sample mean (boxes) and standard deviation (circles) of the deviation of returns from the LML system as function of the number of agents. Different lines correspond to different values of the variance σ_ϵ^2.

7.3.2 LML for Stochastic Dividend

In Section 7.1 we showed that the random dividend yield affects the dynamics of the system but does not destroy the deterministic skeleton prediction. In principle, the matter can change in the presence of the second source of the noise. We explore this question numerically here for the same agents behavior as in the previous section. See Fig. 7.3, where as before two upper panels represent simulations for the system with $F = 0.2$ and two lower panels correspond to the case $F(r) = 0.199 + 0.1\,r$. In all simulations the yield of dividend is i.i.d. variable, so that the LML representation, shown as a thick line, is now stochastic even in the long-run.

Visual impression is that both when the number of agents becomes large and when the variance of the agent-specific noise decreases, the trajectories of the LML become closer to the trajectory of the multi-agent system. In order to obtain some quantitative measure of agreement between the LML and the actual dynamics simulated with both agent-specific and dividend noises, we computed the first two moments of the deviation between these two trajectories. In Fig. 7.4 we report the averages of these moments over 100 independent simulations. It is clear that both an increase in the number of agents and a decrease in the variance of $\epsilon_{t,n}$ lead to the convergence of the many-agent dynamics towards the LML.

In conclusion, we can say that, for what concerns the framework considered in Section 7.2, the LML provides, at least locally, a reasonable approximation of the original models also for a moderately small ($N \sim 50$) population of agents.

7.4 Conclusion

We studied in this Chapter the stochastic versions of the basic model of this part of the thesis. The Large Market Limit introduced above allows to simplify the analysis of the agent-based model with many noisy agents. The LML is a low-dimensional system whose equilibrium and stability analysis can be performed. Through the numerical simulation we showed that the LML provides, at least locally, a reasonable approximation of the original model also for a moderately small ($N \sim 50$) population of agents.

If the LML is, indeed, a good approximation for the dynamics with noisy agents, then the analysis performed in Chapter 5 can be extended. There we gave a complete characterization of the dynamics when arbitrary number of heterogeneous CRRA agents operate in the market. The investment functions of them were "frozen", however, so that the *imitation* of behavior was forbidden. If such imitation took place, the market would end up in the situation like considered in the present Chapter. This fact, and also the general interest to the question of the limits of the "representative" agent metaphor, were the main driving forces of this Chapter.

Appendix

Appendix A

Stability Conditions and Bifurcation Types for 2×2 Matrices

Here we present some classical results from the theory of difference equation. For the extensive theory of the stability and bifurcation analysis of the autonomous dynamical systems see e.g Guckenheimer and Holmes (1983) or Kuznetsov (1995).

Consider a general 2-dimensional non-linear dynamical system

$$\begin{cases} x_{1,t+1} = f_1(x_{1,t}, x_{2,t}) \\ x_{2,t+1} = f_2(x_{1,t}, x_{2,t}) \end{cases} , \tag{A.0.1}$$

and suppose that $(x_1^\star, x_2^\star)$ is a fixed point of the system. Moreover, let $J(x_1^\star, x_2^\star)$ denote the Jacobian matrix of the system (A.0.1) computed in this fixed point:

$$J(x_1^\star, x_2^\star) = \begin{Vmatrix} J_{1,1} & J_{1,2} \\ J_{2,1} & J_{2,2} \end{Vmatrix} . \tag{A.0.2}$$

Let $t = J_{1,1} + J_{2,2}$ and $d = J_{1,1}\,J_{2,2} - J_{1,2}\,J_{2,1}$ be, respectively, the trace and the determinant of matrix (A.0.2). Then the following result takes place:

Proposition A.0.1 (Sufficient conditions for the local stability). *The fixed point* $(x_1^\star, x_2^\star)$ *of the system* (A.0.1) *is locally asymptotically stable if the following conditions are satisfied*

$$d < 1 \quad , \qquad t < 1 + d \quad , \qquad t > -1 - d \quad . \tag{A.0.3}$$

Proof. The characteristic polynomial computed in a fixed point in our notations reads

$$\lambda^2 - t\,\lambda + d = 0$$

Thus, the stability analysis reduces to the analysis of the region in (t, d) parameters space, where the absolute values of both roots

$$\lambda_\pm = \frac{t \pm \sqrt{t^2 - 4\,d}}{2} \tag{A.0.4}$$

are less than 1. There are two cases.

If $t^2 \geq 4\,d$ there are two real roots $\lambda_- \leq \lambda_+$ (coinciding when $t^2 = 4\,d$) and for the local stability we need that both $\lambda_+ < 1$ and also $\lambda_- > -1$. It is straightforward to see that it leads to the following conditions on the trace and determinant:

$$t < 2 \qquad \text{and} \qquad 1 - t + d > 0 \tag{A.0.5}$$

$$t > -2 \qquad \text{and} \qquad 1 + t + d > 0 \quad . \tag{A.0.6}$$

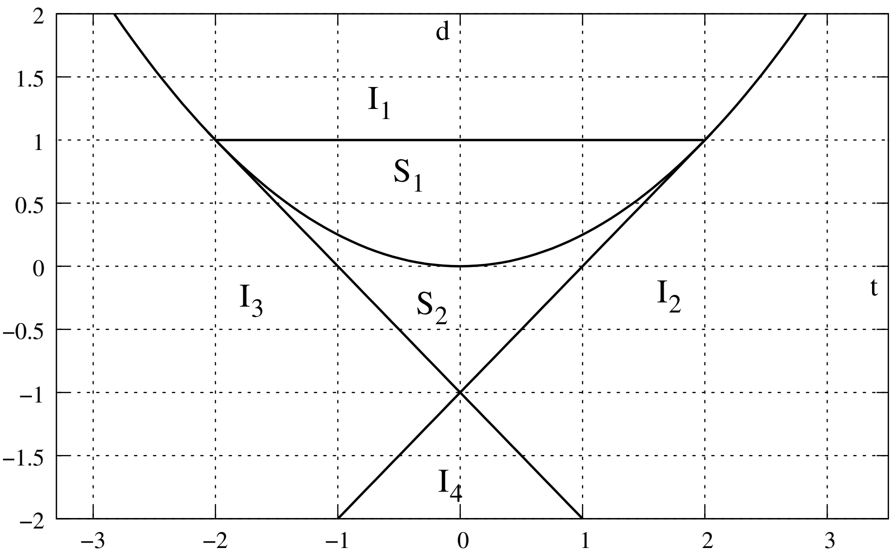

Figure A.1: Stability region for a fixed point of a two-dimensional dynamical system in the Trace-Determinant coordinates.

Area S_1 in Fig. A.1 contains those points where these conditions are satisfied and both roots are real.

Instead, if $t^2 < 4\,d$ there are two complex conjugate roots and the stability condition reads

$$|\lambda| = \sqrt{\lambda_+ \lambda_-} = \sqrt{d} < 1 \tag{A.0.7}$$

Corresponding region is labeled S_2 on Fig. A.1. Now, all points corresponding to stable fixed point lie in the triangle shaped by the union of S_1 and S_2. This triangle is described by conditions (A.0.3). □

Notice that if the fixed point is stable there are two possible qualitative behavior of the system in a neighborhood of it. If eigenvalues of Jacobian are not real, then the local behavior of the system is characterized by the superposition of the rotation and exponential convergence towards the fixed point, so that the resulting time series of both state variables have cyclical component. Such situation is referred as a *focus*. If the eigenvalues of Jacobian are real, then we have a *node*.

According to Proposition A.0.1, if all 3 inequalities in (A.0.3) hold, the fixed point is stable. The fixed point is locally unstable if at least one of that inequalities has an opposite sign. The situation in which with the change of one or more parameters the fixed point loses its stability is called a *bifurcation*. The following Proposition summarizes the information about types of bifurcations.

Proposition A.0.2. *The fixed point of the system* (A.0.1) *looses its stability when one of the inequalities in conditions* (A.0.3) *is changing its sign. Moreover, the system has*

(i) fold bifurcation, if $t = 1 + d$;

(ii) flip bifurcation, if $t = -1 - d$;

(iii) Neimark-Sacker (secondary Hopf) bifurcation, if $d = 1$.

Proof. From the proof of the previous proposition it is clear that when the parameters cross the line separating region S_1 from I_1 at Fig. A.1 (when $d = 1$), the modulus of the complex eigenvalues become larger than 1. Thus the system exhibits a Neimark bifurcation. When the line between S_2 and I_2 is crossed (and so $t = 1 + d$) the largest real eigenvalue becomes greater than 1 a fold bifurcation is observed. Finally, if the line between S_2 and I_3 is crossed (i.e. $t = -1 - d$) the smallest real eigenvalue becomes less than -1, and the flip bifurcation is observed. □

Appendix B

Proofs of Propositions in Chapter 2

B.1 Forecasts of sophisticated fundamentalists

In this Appendix we find the time dynamics of the first two moments of the distribution of return. These moments are used by sophisticated fundamentalists as the forecast for the expected return and variance of the return. In order to do it, we remember that these agents think that the market always tends to correct the current price in the direction to the fundamental value. We can, first, describe such price behavior as mean-reverting stochastic process, then find the first two moments of the price and, finally, go back to the returns.

The following Fokker-Plank equation describes the dynamics of the density of price $f(P, t+\tau)$. Here we denote the time argument as τ to reserve the symbol t for the starting point of the process.

$$\dot{f}(P, t+\tau) = -\frac{\partial}{\partial P}[a_1(P, t+\tau)] + \frac{1}{2}\frac{\partial^2}{\partial P^2}[a_2(P, t+\tau)] \tag{B.1.1}$$

where dot denotes the derivative with respect to time, coefficient of diffusion a_2 is assumed to be constant (we denote it as σ^2), and the coefficient of the drift is assumed to be $\tilde{\theta}(\bar{P} - P)$, where $\tilde{\theta}$ is some positive parameter.

Multiplying both parts of the last equality on P and integrating it with respect to the price, one obtains the differential equation for the first moment of price $m_1(t+\tau) = \int P\, f(P, t+\tau) dP$:

$$\dot{m}_1(t+\tau) = \tilde{\theta}\,\bar{P} - \tilde{\theta}\,m_1(\tau)$$

Solving this equation with the initial condition (when $\tau = 0$) $m_1(t) = P_t$, we get the following

$$m_1(t+\tau) = \bar{P} + (P_t - \bar{P})\,e^{-\tilde{\theta}\tau}$$

Equation (2.3.12) gives the condition for the correct definition of $\tilde{\theta}$, namely we have to impose the restriction $m_1(t+1) = \mathrm{E}_{t,f}[P_{t+1}]$. It reads:

$$\bar{P} + (P_t - \bar{P})\,e^{-\tilde{\theta}} = P_t + \theta\,(\bar{P} - P_t)$$

and so

$$\tilde{\theta} = -\ln(1-\theta) \tag{B.1.2}$$

If we compute now the first moment m_1 for the integer $\tau = \eta$, we get the belief of the agent about the average price. Since $\mathrm{E}_t[\rho_{t+\eta}] = \mathrm{E}_t[P_{t+\eta}]/P_t - 1$ we can obtain the belief about the average of the return: $\mathrm{E}_t[\rho_{t+\eta}] = (\bar{P}/P_t - 1)(1 - e^{-\tilde{\theta}\eta})$. Finally changing $\tilde{\theta}$ according to equation (B.1.2) we get equation (2.3.13).

Multiplying both parts of (B.1.1) on P^2 and integrating the result with respect to the price, we get the differential equation for the second moment of price $m_2(t+\tau) = \int P^2 f(P, t+\tau)\, dP$:

$$\dot{m}_2(t+\tau) = \sigma^2 + 2\tilde{\theta}\,\bar{P}\,m_1(t+\tau) - 2\,\tilde{\theta}\,m_2(t+\tau)$$

The solution of that equation with the initial value $m_2(t) = P_t^2$ (which corresponds to zero variance for $\tau = 0$) reads:

$$m_2(t+\tau) = \frac{\sigma^2}{2\,\tilde{\theta}} + \bar{P}^2 + 2\bar{P}(P_t - \bar{P})e^{-\tilde{\theta}\tau} + \left(P_t^2 - \frac{\sigma^2}{2\,\tilde{\theta}} - \bar{P}^2 - 2\,\bar{P}\,(P_t - \bar{P})\right) e^{-2\tilde{\theta}\tau}$$

Now using the expression for the first two moments of price distribution, we can compute the belief of the fundamentalists about the variance $\mathrm{V}_{t,f}[\rho_{t,t+\eta}]$ as a function of parameters:

$$\mathrm{V}_{t,f}[\rho_{t,t+\eta}] = \frac{1}{P_t^2}\left(m_2(t+\eta) - m_1(t+\eta)^2\right) = \frac{\sigma^2}{2\tilde{\theta}P_t^2}\left(1 - e^{-2\tilde{\theta}\eta}\right)$$

To get rid of the parameter σ^2 we impose the condition that making one period forecast fundamentalists behave like chartists, i.e. that for $\eta = 1$ the last expression coincides with the EWMA forecast z_{t-1}. Then

$$\sigma^2 = \frac{2\tilde{\theta}P_t^2}{1 - e^{-2\tilde{\theta}}}\, z_{t-1}$$

and, finally,

$$\mathrm{V}_{t,f}[\rho_{t,t+\eta}] = \frac{1 - e^{-2\tilde{\theta}\eta}}{1 - e^{-2\tilde{\theta}}}\, z_{t-1}$$

Using now equation (B.1.2) to express $\tilde{\theta}$ through θ, we get (2.3.15).

Finally notice, that if we start with Fokker-Plank equation (B.1.1) where both diffusion and drift coefficients are constant, then using the same technique we get the forecasted rules (2.3.10) and (2.3.11) for chartists.

B.2 Analysis of System (2.4.2)

Notice, first of all, that even if function f in (2.4.2) is defined only for positive arguments, it can be extended continuously for $z = 0$. In fact,

$$\lim_{z\to 0} f(z) = \frac{s}{r} = \gamma\frac{d}{r} = \gamma\bar{P}$$

Thus, system (2.4.2) becomes to be defined for any y and for $z \geq 0$. With such extension the system has only one fixed point: $(\gamma\bar{P}, 0, 0)$, which corresponds to the fundamental price with zero forecasted variance.

The local stability of the point can be checked computing the Jacobian matrix. First, note that the derivative of function f reads

$$f'(z) = \frac{1}{z}\left(\frac{s}{\sqrt{r^2 + 4sz}} - f(z)\right) \quad .$$

This derivative can be extended to the point $z = 0$ continuously, so that $f'(0) = -s^2/r^3$. Second, we write the Jacobian matrix in $(\gamma\bar{P}, 0, 0)$:

$$\mathbf{J}(p, y, z)\Big|_{(\gamma\bar{P},0,0)} = \left\|\begin{array}{ccc} 0 & 0 & f'(0) \\ -(1-\lambda)\frac{f(0)}{(\gamma\bar{P})^2} & \lambda & (1-\lambda)\frac{f'(0)}{\gamma\bar{P}} \\ -2\lambda(1-\lambda)\frac{f(0)}{(\gamma\bar{P})^2}\tilde{h} & -2\lambda(1-\lambda)\tilde{h} & \lambda + 2\lambda(1-\lambda)\tilde{h}\frac{f'(0)}{(\gamma\bar{P})} \end{array}\right\| \quad ,$$

where $\tilde{h}$ stands for the value of the function $h(p, y, z) = \frac{f(z)}{p} - 1 - y$ in point $(\gamma\bar{P}, 0, 0)$. Since $\tilde{h} = 0$, the Jacobian in the fixed point can be simplified as follows:

$$\mathbf{J}(p, y, z)\Big|_{(\gamma\bar{P},0,0)} = \begin{Vmatrix} 0 & 0 & -s^2/r^3 \\ -(1-\lambda)r/s & \lambda & -(1-\lambda)s/r^2 \\ 0 & 0 & 0 \end{Vmatrix} .$$

It is obvious that eigenvalues of $\mathbf{J}(p, y, z)$ are λ and 0 (with multiplicity 2, since the trace is λ), which implies that the point $(\gamma\bar{P}, 0, 0)$ of system (2.4.2) is locally asymptotically stable as far as $0 \le \lambda < 1$.

B.3 Analysis of System (2.5.1)

To analyze the stability of the system, we, first, note that function g can be extended continuously to $z = 0$ when $y < r$. The same is true for its derivatives. Namely, since

$$\begin{aligned} g'_y(y, z) &= g(y, z)/\sqrt{(y-r)^2 + 4sz} \\ g'_z(y, z) &= (s/\sqrt{(y-r)^2 + 4sz} - g(y, z))/z \end{aligned}$$

we can define

$$\begin{aligned} g(0, 0) &= s/r \\ g'_y(0, 0) &= s/r^2 \\ g'_z(0, 0) &= -s^2/r^3 \end{aligned}$$

Then the Jacobian of the system reads:

$$\mathbf{J}(x, y, z) = \begin{Vmatrix} 0 & g'_y & g'_z \\ -(1-\lambda)\frac{g}{x^2} & \lambda + (1-\lambda)\frac{g'_y}{x} & (1-\lambda)\frac{g'_z}{x} \\ -2\lambda(1-\lambda)h\frac{g}{x^2} & 2\lambda(1-\lambda)h\left(\frac{g'_z}{x} - 1\right) & \lambda + 2\lambda(1-\lambda)h\frac{g'_z}{x} \end{Vmatrix}$$

where $h(x, y, z) = g(y, z)/x - 1 - y$.

Bottazzi (2002) showed that for generic function g the eigenvalues of this Jacobian (and so of the system (2.5.1)) computed in the fixed point depend only on two parameters: λ and $a = \partial_y ln(g(0, 0))$. In the case of our function $g(y, z)$, we have $a = 1/r$ and three eigenvalues read:

$$\begin{aligned} \mu_0 &= \lambda \\ \mu_1 &= \tfrac{1}{2}\Big(\lambda + (1-\lambda)\tfrac{1}{r} + \sqrt{(\lambda + (1-\lambda)\tfrac{1}{r})^2 - 4(1-\lambda)\tfrac{1}{r}}\Big) \\ \mu_2 &= \tfrac{1}{2}\Big(\lambda + (1-\lambda)\tfrac{1}{r} - \sqrt{(\lambda + (1-\lambda)\tfrac{1}{r})^2 - 4(1-\lambda)\tfrac{1}{r}}\Big) \end{aligned}$$

It is easy to show that μ_1 and μ_2 have modulus less than 1 *iff* $\lambda > 1 - r$, and that these eigenvalues cross the unit circle being complex.

Appendix C

Proofs of Propositions in Chapter 3

C.1 Proof of Proposition 3.6.1

Plugging the expression for $w_{t+1,n}$ from the second equation in system (3.5.9) into the right-hand side of the first equation of the same system, and assuming that $p_t > 0$ and, consistently with (3.6.3), $p_t \neq \sum x_{t+1,n}\, x_{t,n}\, w_{t,n}$ one gets

$$
\begin{aligned}
p_{t+1} &= \left(1 - \frac{1}{p_t}\sum_{n=1}^{N} x_{t+1,n}\, x_{t,n}\, w_{t,n}\right)^{-1} \left(\sum_{n=1}^{N} x_{t+1,n}\, w_{t,n} + \big(e_{t+1} - 1\big)\sum_{n=1}^{N} x_{t+1,n}\, w_{t,n}\, x_{t,n}\right) = \\
&= p_t\, \frac{\sum_n x_{t+1,n}\, w_{t,n} + (e_{t+1}-1)\sum_n x_{t+1,n}\, w_{t,n}\, x_{t,n}}{\sum_n x_{t,n}\, w_{t,n} - \sum_n x_{t+1,n}\, x_{t,n}\, w_{t,n}} = \\
&= p_t\, \frac{\big\langle x_{t+1}\big\rangle_t - \big\langle x_t\, x_{t+1}\big\rangle_t + e_{t+1}\big\langle x_t\, x_{t+1}\big\rangle_t}{\big\langle x_t\big\rangle_t - \big\langle x_t\, x_{t+1}\big\rangle_t} \quad ,
\end{aligned}
$$

where we used the first equation of (3.5.9) rewritten for time t to get the second equality. Condition (3.6.3) is obtained imposing $p_{t+1} > 0$, and the dynamics of price return in (3.6.4) is immediately derived. From the second equation of (3.5.9) it follows that

$$
w_{t+1,n} = w_{t,n}\,\big(1 + x_{t,n}\,(r_{t+1} + e_{t+1})\big) \qquad \forall n \in \{1,\ldots,N\} \quad , \tag{C.1.1}
$$

which leads to (3.6.5). To obtain the wealth share dynamics, divide both sides of (C.1.1) by w_{t+1} to have

$$
\begin{aligned}
\varphi_{t+1,n} &= \frac{w_{t,n}}{\sum_m w_{t+1,m}}\,\big(1 + x_{t,n}\,(r_{t+1} + e_{t+1})\big) = \\
&= \frac{w_{t,n}}{\sum_m w_{t,m} + (r_{t+1} + e_{t+1})\sum_m x_{t,m} w_{t,m}}\,\big(1 + x_{t,n}\,(r_{t+1} + e_{t+1})\big) = \\
&= \frac{\varphi_{t,n}}{1 + (r_{t+1} + e_{t+1})\sum_m x_{t,m}\varphi_{t,m}}\,\big(1 + x_{t,n}\,(r_{t+1} + e_{t+1})\big) \quad ,
\end{aligned}
$$

where (C.1.1) has been used to get the second line and we divided both numerator and denominator by the total wealth at time t to get the third.

C.2 Proof of Proposition 3.6.2

If (3.6.8) is valid for all the x's it is also valid for the wealth-weighted averages, i.e. we have:

$$\begin{array}{rcccl} x_{\min} & \leq & \langle x_t \rangle & \leq & x_{\max} \quad , \\ x_{\min}^2 & \leq & \langle x_t \, x_{t+1} \rangle & \leq & x_{\max}^2 \quad , \\ x_{\min} \, (1 - x_{\max}) & \leq & \langle x_t \, (1 - x_{t+1}) \rangle & \leq & x_{\max} \, (1 - x_{\min}) \quad . \end{array}$$

Since $e_{t+1} > 0$, both factors on the left hand side of (3.6.3) are positive and so the constraint is satisfied. At the same time, since the denominator of the expressions in (3.6.5) and (3.6.4) is strictly greater than zero, and numerators are bounded, these two expressions remain bounded.

Appendix D

Proofs of Propositions in Chapter 4

D.1 Proof of Proposition 4.3.1

Item *(i)* is obvious after direct substitution of the equilibrium values in system (4.3.2). Item *(ii)* is equivalent to (3.6.3) rewritten in the equilibrium. Item *(iii)* follows from (3.6.5) and the results in item *(i)*:

$$\rho^* = x^* \, (r^* + \bar{e}) = l(r^*) \, (r^* + \bar{e}) = r^* \quad .$$

D.2 Proof of Proposition 4.3.2

The Jacobian matrix J of the system in a fixed point reads

$$J = \begin{Vmatrix} 0 & f' \\ -1/(x^* \, (1-x^*)) & (1+r^*) \, f'/(x^* \, (1-x^*)) \end{Vmatrix} \quad ,$$

and has the following characteristic polynomial

$$\mu^2 - \mu \left(f' \, \frac{1+r^*}{x^*(1-x^*)} \right) + f' \, \frac{1}{x^*(1-x^*)} \quad . \tag{D.2.1}$$

The statement of the proposition can now be obtained, applying Proposition A.0.1 from Appendix A and using the following relation:

$$l'(r^*) = \frac{\bar{e}}{(r^* + \bar{e})^2} = \frac{x^* \, (1-x^*)}{r^*} \quad . \tag{D.2.2}$$

The second equality can be directly checked from (4.3.3). Thus, three inequalities (4.3.4) represent the conditions for both the roots of the polynomial (D.2.1) to be inside the unit circle.

The last part of the proposition directly follows from Proposition A.0.2 of Appendix A.

D.3 Proof of Proposition 4.3.4

The Jacobian matrix J of the system in a fixed point reads

$$J = \begin{Vmatrix} 0 & f' \\ -(1-\lambda)/(x^* \, (1-x^*)) & \lambda + (1-\lambda) \, (1+r^*) \, f'/(x^* \, (1-x^*)) \end{Vmatrix} \quad ,$$

and has the following characteristic polynomial

$$\mu^2 - \mu \left(\lambda + (1-\lambda)\, f' \, \frac{1+r^*}{x^*(1-x^*)} \right) + (1-\lambda)\, f' \, \frac{1}{x^*(1-x^*)} \quad . \tag{D.3.1}$$

Applying the result of Proposition A.0.1 from Appendix A and using (D.2.2), the statement is obtained. Three inequalities (4.3.7) represent the conditions for both the roots of the polynomial (D.3.1) to be inside the unit circle.

The last part of the proposition directly follows from Proposition A.0.2 of Appendix A.

D.4 Proof of Proposition 4.3.6

The Jacobian matrix of the system in a fixed point reads

$$\begin{Vmatrix} 0 & f'_y & f'_z \\ -(1-\lambda)/a & \lambda + (1-\lambda)\,(1+y^*)\, f'_y/a & (1-\lambda)\,(1+y^*)\, f'_z/a \\ 0 & 0 & \lambda \end{Vmatrix} ,$$

where $a = x^*\,(1-x^*)$ and f'_z is the partial derivative of the function f with respect to the second variable computed at the equilibrium. One of the eigenvalues is $\lambda < 1$, while the others are the roots of the polynomial (D.3.1) with f' replaced by f'_y. The statement immediately follows.

D.5 Proof of Proposition 4.4.1

Since $\lambda_n \neq 1$ for all n, the first set of equalities in (4.4.4) follows immediately from block $\mathcal{Y}$. Then, the second part of (4.4.4) is also obvious from the equations in block $\mathcal{Z}$. Plugging the resulting relations into the equations of block $\mathcal{X}$ one has (4.4.7) and (4.4.10).

From block $\mathcal{W}$ using (4.4.2) and the condition $r^* + \bar{e} \neq 0$ one obtains

$$\varphi^*_n = 0 \quad \text{or} \quad \sum_{m=1}^{N-1} \varphi^*_m \, x^*_m + \left(1 - \sum_{m=1}^{N-1} \varphi^*_m \right) x^*_N = x^*_n \quad \forall n \in \{1, \dots, N-1\} \quad . \tag{D.5.1}$$

Finally, the relation (4.4.3) in equilibrium reads:

$$r^* = \bar{e}\, \frac{\sum_{n=1}^{N-1} \varphi^*_n \, {x^*_n}^2 + \left(1 - \sum_{n=1}^{N-1} \varphi^*_n\right) {x^*_N}^2}{\sum_{n=1}^{N-1} \varphi^*_n \, x^*_n \,(1-x^*_n) + \left(1 - \sum_{n=1}^{N-1} \varphi^*_n\right) x^*_N \,(1-x^*_N)} \quad . \tag{D.5.2}$$

This set of equations admits two types of solutions, depending on how many equilibrium wealth shares are different from zero: if one or many.

To derive the first type of solutions assume (4.4.5). In this case (D.5.1) is, obviously, automatically satisfied for all agents. From (D.5.2) one has $x^*_1 = r^*/(\bar{e} + r^*)$ which together with (4.4.7) leads to (4.4.6).

To derive the second type of solutions assume (4.4.8). In this case, the second equality of (D.5.1) must be satisfied for any $n \leq k$. Since its left-hand side does not depend on n, a $x^*_{1\diamond k}$ must exist such that $x^*_1 = \cdots = x^*_k = x^*_{1\diamond k}$. Substituting $x^*_n = 0$ for $n > k$ and $x^*_n = x^*_{1\diamond k}$ for $n \leq k$ in (D.5.2) one gets $x^*_{1\diamond k} = r^*/(\bar{e} + r^*)$. The equilibrium return r^* is implicitly defined combining this last relation with (4.4.10) for $n \leq k$.

The equilibrium wealth growth rate of the survivors is immediately obtained from (3.6.5) and from (4.4.7) or (4.4.10) for the single survivor and the many survivors case, respectively.

D.6 Proof of Corollary 4.4.2

We, first, consider the case $N = 2$ and suppose that the investment shares in $-\bar{e}$ are such that $x_1^* < 0$ and $x_2^* > 0$. Then condition (4.4.11) implies that

$$\varphi_1^* = \frac{x_2^*}{x_2^* - x_1^*} .$$

Since $\varphi_1^* \in (0, 1)$, the equilibrium exists and is unique.

It is now obvious that if requirement (4.4.13) is satisfied, then "no arbitrage" equilibrium does exist. Let us suppose that there is no couple of agents with positive and negative investment shares. In this case, in order to satisfy (4.4.11), all agents with non-zero wealth shares φ's have to invest zero in the risky asset. But then we have $\sum \varphi_n^* {x_n^*}^2 = 0$ and condition (3.6.3) is not satisfied.

The remaining part of the corollary follows from (4.4.11) straight-forwardly.

D.7 Proofs of Propositions in Section 4.4.3

Before proving Propositions 4.4.3, 4.4.4 and 4.4.5 we need some preliminary results. The Jacobian matrix of the deterministic skeleton of system (4.4.1) is a $\big(4N-1\big) \times \big(4N-1\big)$ matrix. Using the block structure introduced in Section 4.4.1 it can be separated in 16 blocks

$$\boldsymbol{J} = \begin{Vmatrix} \frac{\partial \mathcal{X}}{\partial \mathcal{X}} & \frac{\partial \mathcal{X}}{\partial \mathcal{Y}} & \frac{\partial \mathcal{X}}{\partial \mathcal{Z}} & \frac{\partial \mathcal{X}}{\partial \mathcal{W}} \\ \frac{\partial \mathcal{Y}}{\partial \mathcal{X}} & \frac{\partial \mathcal{Y}}{\partial \mathcal{Y}} & \frac{\partial \mathcal{Y}}{\partial \mathcal{Z}} & \frac{\partial \mathcal{Y}}{\partial \mathcal{W}} \\ \frac{\partial \mathcal{Z}}{\partial \mathcal{X}} & \frac{\partial \mathcal{Z}}{\partial \mathcal{Y}} & \frac{\partial \mathcal{Z}}{\partial \mathcal{Z}} & \frac{\partial \mathcal{Z}}{\partial \mathcal{W}} \\ \frac{\partial \mathcal{W}}{\partial \mathcal{X}} & \frac{\partial \mathcal{W}}{\partial \mathcal{Y}} & \frac{\partial \mathcal{W}}{\partial \mathcal{Z}} & \frac{\partial \mathcal{W}}{\partial \mathcal{W}} \end{Vmatrix} . \tag{D.7.1}$$

The block $\partial\mathcal{X}/\partial\mathcal{X}$ is a $N \times N$ matrix containing the partial derivatives of the agents' present investment choices with respect to the agents' past investment choices. According to (4.2.1) the investment choice of any agent does not explicitly depend on the investment choices in previous period, therefore,

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{X}}\right]_{n,m} = \frac{\partial f_n}{\partial x_m} = 0 \, , \qquad 1 \leq n, m \leq N \quad ,$$

and this block is a zero matrix.

The block $\partial\mathcal{X}/\partial\mathcal{Y}$ is a $N \times N$ matrix containing the partial derivatives of the agents' investment choices with respect to the agents' return forecasts. Since the choice of any agent does not depend on the forecasts of other agents, this block is a diagonal matrix with diagonal elements

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{Y}}\right]_{n,n} = \frac{\partial f_n}{\partial y_n} = f'_{n,y} \, , \qquad 1 \leq n \leq N \quad .$$

Analogously the $N \times N$ block $\partial\mathcal{X}/\partial\mathcal{Z}$ of the partial derivatives of the agents' investment choices with respect to the agents' variance forecasts is a diagonal matrix with diagonal elements

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{Z}}\right]_{n,n} = \frac{\partial f_n}{\partial z_n} = f'_{n,z} \, , \qquad 1 \leq n \leq N \quad .$$

The block $\partial\mathcal{X}/\partial\mathcal{W}$ is a $N \times (N-1)$ matrix containing the partial derivatives of the agents' investment choices with respect to the agents' investment shares. Under Assumption $2'$ this is a zero matrix

$$\left[\frac{\partial \mathcal{X}}{\partial \mathcal{W}}\right]_{n,m} = \frac{\partial f_n}{\partial \varphi_m} = 0 \, , \qquad 1 \leq n \leq N, \quad 1 \leq m \leq N-1 \quad .$$

The definitions of the next blocks will make use of the function in the right-hand side of (4.4.3) which gives the evolution of return. This function depends on the agents' previous investment choices $x_{t,n}$, the agents' wealth shares $\varphi_{t,n}$ and the agents' contemporaneous investment choices given by the investment functions f_n for $n \in \{1, \dots, N\}$. We denote the corresponding derivatives as r'_{x_n}, r'_{φ_n} and r'_{f_n}.

The block $\partial\mathcal{Y}/\partial\mathcal{X}$ is a $N \times N$ matrix containing the partial derivatives of the agents' return forecasts with respect to the agents' investment shares. The elements of this block read:

$$\left[\frac{\partial\mathcal{Y}}{\partial\mathcal{X}}\right]_{n,m} = \frac{\partial y_n}{\partial x_m} = (1-\lambda_n)\, r'_{x_m} \,, \qquad 1 \le n, m \le N \,.$$

The block $\partial\mathcal{Y}/\partial\mathcal{Y}$ is a $N \times N$ matrix containing the partial derivatives of the agents' return forecasts with respect to the same set of variables. Using the chain rule one can easily check that this block contains the following elements:

$$\left[\frac{\partial\mathcal{Y}}{\partial\mathcal{Y}}\right]_{n,m} = \frac{\partial y_n}{\partial y_m} = \lambda_n\, \delta_{n,m} + (1-\lambda_n)\, r'_{f_m}\, f'_{m,y} \,, \qquad 1 \le n, m \le N \,,$$

where $\delta_{n,m}$ stands for the Kronecker delta.

The block $\partial\mathcal{Y}/\partial\mathcal{Z}$ is a $N \times N$ matrix containing the partial derivatives of the agents' return forecasts with respect to the agents' forecasts for variance. Using the chain rule we find that the elements of this block read:

$$\left[\frac{\partial\mathcal{Y}}{\partial\mathcal{Z}}\right]_{n,m} = \frac{\partial y_n}{\partial z_m} = (1-\lambda_n)\, r'_{f_m}\, f'_{m,z} \,, \qquad 1 \le n, m \le N \,.$$

The block $\partial\mathcal{Y}/\partial\mathcal{W}$ is a $N \times (N-1)$ matrix containing the partial derivatives of the agents' return forecasts with respect to the agents' wealth shares. The elements of this block are:

$$\left[\frac{\partial\mathcal{Y}}{\partial\mathcal{W}}\right]_{n,m} = \frac{\partial y_n}{\partial \varphi_m} = (1-\lambda_n)\, r'_{\varphi_m} \,, \qquad 1 \le n \le N, \quad 1 \le m \le N-1 \,.$$

The block $\partial\mathcal{Z}/\partial\mathcal{X}$ is a $N \times N$ matrix containing the partial derivatives of the agents' variance forecasts with respect to the agents' investment shares. In any equilibrium this block is a zero matrix:

$$\left[\frac{\partial\mathcal{Z}}{\partial\mathcal{X}}\right]_{n,m} = \frac{\partial z_n}{\partial x_m} = 0 \,, \qquad 1 \le n, m \le N \,.$$

Indeed, the derivative $\partial z_n/\partial x_m$ contains factor $r_{t+1} - y_{t,n}$, which is annulated in any equilibrium due to (4.4.4).

Analogously, the block $\partial\mathcal{Z}/\partial\mathcal{Y}$ which is a $N \times N$ matrix containing the partial derivatives of the agents' variance forecasts with respect to the agents' return forecasts, is a zero block in equilibrium:

$$\left[\frac{\partial\mathcal{Z}}{\partial\mathcal{Y}}\right]_{n,m} = \frac{\partial z_n}{\partial y_m} = 0 \,, \qquad 1 \le n, m \le N \,.$$

The same reasoning allows to simplify the $N \times N$ block $\partial\mathcal{Z}/\partial\mathcal{Z}$ containing the partial derivatives of the agents' variance forecasts with respect to themselves. In equilibrium this is a diagonal matrix with diagonal elements

$$\left[\frac{\partial\mathcal{Z}}{\partial\mathcal{Z}}\right]_{n,n} = \frac{\partial z_n}{\partial z_n} = \lambda_n \,, \qquad 1 \le n \le N \quad .$$

The block $\partial\mathcal{Z}/\partial\mathcal{W}$ is a $N \times (N-1)$ matrix containing the partial derivatives of the agents' variance forecasts with respect to the agents' wealth shares. Again, in any equilibrium this is a zero block:

$$\left[\frac{\partial\mathcal{Z}}{\partial\mathcal{W}}\right]_{n,m} = \frac{\partial z_n}{\partial \varphi_m} = 0 \,, \qquad 1 \le n \le N, \quad 1 \le m \le N-1 \,.$$

The block $\partial\mathcal{W}/\partial\mathcal{X}$ is a $(N-1)\times N$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' investment shares. It is:

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{X}}\right]_{n,m} = \frac{\partial\varphi_n}{\partial x_m} = \Phi_n^{x_m} + \Phi_n^r\cdot r'_{x_m}\,, \qquad 1\le n\le N-1\,,\quad 1\le m\le N\,,$$

where $\Phi_n^{x_m} = \partial\Phi_n/\partial x_m$ and $\Phi_n^r = \partial\Phi_n/\partial r$.

The block $\partial\mathcal{W}/\partial\mathcal{Y}$ is a $(N-1)\times N$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' return forecast. It is

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{Y}}\right]_{n,m} = \frac{\partial\varphi_n}{\partial y_m} = \Phi_n^r\cdot r'_{f_m}\cdot f'_{m,y}\,, \qquad 1\le n\le N-1\,,\quad 1\le m\le N\,.$$

The block $\partial\mathcal{W}/\partial\mathcal{Z}$ is a $(N-1)\times N$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' variance forecasts. It is

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{Z}}\right]_{n,m} = \frac{\partial\varphi_n}{\partial z_m} = \Phi_n^r\cdot r'_{f_m}\cdot f'_{m,z}\,, \qquad 1\le n\le N-1\,,\quad 1\le m\le N\,.$$

The block $\partial\mathcal{W}/\partial\mathcal{W}$ is a $(N-1)\times(N-1)$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' wealth shares. The elements of this block are:

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{W}}\right]_{n,m} = \frac{\partial\varphi_n}{\partial\varphi_m} = \Phi_n^{\varphi_m} + \Phi_n^r\cdot r'_{\varphi_m}\,, \qquad 1\le n,m\le N-1\,,$$

where $\Phi_n^{\varphi_m} = \partial\Phi_n/\partial\varphi_m$.

Our next steps consist in the characterization of the Jacobian matrix (D.7.1) in the fixed point, computation of the characteristic polynomial and, finally, analysis of its roots. Since there are two general types of the equilibria, we separate our further analysis. First, we consider the case with $r^*\neq-\bar{e}$ and prove Propositions 4.4.3 and 4.4.4. Then, we move to the case where $r^*=-\bar{e}$ and eventually prove Proposition 4.4.5.

D.7.1 Equilibria with $r^*\neq-\bar{e}$.

Let $x_{1\diamond k}$ denote the equilibrium investment shares of survivor(s) in such equilibrium. The Jacobian structure in the fixed point is determined by the values of derivatives of functions providing wealth shares and return. We compute them in the following

Lemma D.7.1. *Consider equilibrium $\boldsymbol{x^*}$ of system* (4.4.1) *with $k\ge1$ survivors and $r^*\neq-\bar{e}$. In this equilibrium functions Φ_n defined in* (4.4.2) *for all $n\in\{1,\dots,N-1\}$ have the following derivatives:*

$$\begin{aligned}
\Phi_n^{x_m} &= \varphi_n^*\left(\delta_{n,m}-\varphi_m^*\right)\frac{\bar{e}+r^*}{1+r^*} && \forall m\in\{1,\dots,N\}\quad, \\
\Phi_n^{\varphi_m} &= \frac{\delta_{n,m}\big(1+x_n^*(r^*+\bar{e})\big)-\varphi_n^*(r^*+\bar{e})(x_m^*-x_N^*)}{1+r^*} && \forall m\in\{1,\dots,N-1\}\quad, \\
\Phi_n^r &= \varphi_n^*\,\frac{x_n^*-x_{1\diamond k}^*}{1+r^*}\quad,
\end{aligned} \tag{D.7.2}$$

where $\delta_{n,m}$ is the Kronecker delta. Furthermore, the expression in the right-hand side of (4.4.3)*, providing the price return, has the following derivatives in this equilibrium*

$$\begin{aligned}
r'_{x_m} &= -\,\varphi_m^*\,\frac{1}{x_{1\diamond k}^*\,(1-x_{1\diamond k}^*)} && \forall m\in\{1,\dots,N\}\quad, \\
r'_{f_m} &= \varphi_m^*\,\frac{1+r^*}{x_{1\diamond k}^*\,(1-x_{1\diamond k}^*)} && \forall m\in\{1,\dots,N\}\quad, \\
r'_{\varphi_m} &= \frac{r^*+\bar{e}}{x_{1\diamond k}^*\,(1-x_{1\diamond k}^*)}\,\left(x_m^*\,(x_m^*-x_{1\diamond k}^*)-x_N^*\,(x_N^*-x_{1\diamond k}^*)\right) && \forall m\in\{1,\dots,N-1\}\,.
\end{aligned} \tag{D.7.3}$$

This statement can be checked with the direct computations. The main simplifications come from two properties of the fixed point which we found in Proposition 4.4.1. First, in any fixed point for any agent n it is either $\varphi_n^* = 0$ or $x_n^* = x_{1\diamond k}^*$. Second, from (4.4.9) it follows that $x_{1\diamond k}^* \left(r^* + \bar{e}\right) = r^*$.

With results of Lemma D.7.1 one can simplify the Jacobian (D.7.1). First, since all the survivors invest the same share at the equilibrium, for any agent n it is $\Phi_n^r = 0$. Blocks $[\partial\mathcal{W}/\partial\mathcal{Y}]$ and $[\partial\mathcal{W}/\partial\mathcal{Z}]$ are zero matrices, therefore. Second, if agent n does not survive, i.e. $\varphi_n^* = 0$, then $\Phi_n^{x_m} = 0$. If, instead, agent n survives but agent m does not survive, i.e. $\varphi_m^* = 0$, then again $\Phi_n^{x_m} = 0$. It implies

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{X}}\right]_{n,m} = \begin{cases} \Phi_n^{x_m} & m, n \leq k \\ 0 & \text{otherwise} \end{cases} .$$

Third, also if agent n does not survive then, for any other agent m it is $\Phi_n^{\varphi_m} = 0$, and therefore

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{W}}\right]_{n,m} = \begin{cases} 0 & n > k\,, \quad n \neq m \\ \Phi_n^{\varphi_m} & \text{otherwise} \end{cases} .$$

These findings substantially simplify the last row of blocks of (D.7.1).

Finally, from (D.7.3) it follows that $r'_{x_m} = r'_{f,m} = 0$ for $m > k$, i.e. for those who do not survive. This leads to simplifications in the second row of blocks of (D.7.1).

Summing up all simplifications, we get the following

Lemma D.7.2. *Let $\boldsymbol{x}^*$ be an equilibrium of system (4.4.1) with $k \geq 1$ survivors and $r^* \neq -\bar{e}$. The Jacobian matrix computed in this point $\boldsymbol{J}(\boldsymbol{x}^*)$ has the structure of the following matrix, where each one of the 16 blocks defined above is divided into 4 sub-blocks to display the upper-left $k \times k$ minor matrix. All non-zero elements are denoted by the same symbol "$\star$".*

$$\left\| \begin{array}{cccccc|cccccc|cccccc|cccccc}
0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \star & 0 & \dots & 0 & 0 & \dots & \star & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \star & 0 & \dots & 0 & 0 & \dots & \star & 0 & \dots & 0 & 0 & \dots & 0 \\
\hline
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star \\
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & 0 & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & 0 & \dots & \star & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star \\
\hline
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \star & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \star & 0 & \dots & 0 & 0 & \dots & 0 \\
\hline
\star & \dots & \star & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\star & \dots & \star & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \star
\end{array} \right\|$$

Lemma D.7.3. *The characteristic polynomial P_J of the matrix $\boldsymbol{J}(\boldsymbol{x}^*)$ reads*

$$P_J(\mu) = (-1)^N \, \mu^{N-1} \, (1-\mu)^{k-1} \prod_{j=k+1}^{N} \left(\frac{1 + x_j^* \, (r^* + \bar{e})}{1 + r^*} - \mu \right) \prod_{j=1}^{k} (\lambda_j - \mu) \prod_{j=k+1}^{N} (\lambda_j - \mu)^2$$
$$\left(\mu \prod_{j=1}^{k} (\lambda_j - \mu) + \frac{(1+r^*)\mu - 1}{x_{1\diamond k}^* (1 - x_{1\diamond k}^*)} \sum_{j=1}^{k} \left(\varphi_j^* \, f'_{j,y} (1 - \lambda_j) \prod_{i=1, i \neq j}^{k} (\lambda_i - \mu) \right) \right) \tag{D.7.4}$$

Proof. The following proof is constructive: we will identify in succession the factors appearing in (D.7.4). At each step, a set of eigenvalues is found and the problem is reduced to the analysis of the residual matrix obtained removing the rows and columns associated with the relative eigenspace. In this way the dimension of the analyzed matrix is progressively reduced.

Consider the Jacobian matrix in Lemma D.7.2. One can easily see that in each of the N rows belonging to the third row of blocks (block rows are separated by single lines) the only non-zero entries are the λ's on the diagonal of $[\partial\mathcal{Z}/\partial\mathcal{Z}]$. Consequently, $\lambda_1, \ldots, \lambda_N$ are eigenvalues of the matrix, with multiplicity (at least) one. Also in each of the last $N-1-k$ rows of the matrix, the only non-zero entries belong to the main diagonal of $[\partial\mathcal{W}/\partial\mathcal{W}]$. Thus, when $k < N-1$, we have the $N-1-k$ eigenvalues $\Phi_n^{\varphi_n}$ for $k+1 \leq n \leq N-1$, computed in (D.7.2). It is also obvious that the last $N-k$ columns of the leftmost blocks contain only zero entries so that the matrix possesses eigenvalue 0 with (at least) multiplicity $N-k$. A first contribution to the characteristic polynomial is then determined as

$$(-\mu)^{N-k} \prod_{j=1}^{N} (\lambda_j - \mu) \prod_{j=k+1}^{N-1} (\Phi_j^{\varphi_j} - \mu) =$$
$$= (-\mu)^{N-k} \prod_{j=1}^{N} (\lambda_j - \mu) \prod_{j=k+1}^{N-1} \left(\frac{1 + x_j^*(r^* + \bar{e})}{1 + r^*} - \mu \right) \quad . \tag{D.7.5}$$

In order to find the remaining factors we eliminate the rows and columns associated to the previous eigenvalues. Consider now the last $N-k$ columns in the second blocks column in the remaining matrix. The only non zero elements are on the main diagonal of $[\partial\mathcal{Y}/\partial\mathcal{Y}]$. This leads, if $k < N$, to a second contribution

$$\prod_{j=k+1}^{N} (\lambda_j - \mu) \quad . \tag{D.7.6}$$

After the corresponding further elimination of rows and columns the following matrix is obtained

$$\left\| \begin{array}{ccc|ccc|ccc}
0 & \cdots & 0 & f'_{1,y} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & f'_{k,y} & 0 & \cdots & 0 \\
\hline
(1-\lambda_1)r'_{x_1} & \cdots & (1-\lambda_1)r'_{x_k} & \lambda_1 + (1-\lambda_1)r'_{f_1} f'_{1,y} & \cdots & (1-\lambda_1)r'_{f_k} f'_{k,y} & (1-\lambda_1)r'_{\varphi_1} & \cdots & (1-\lambda_1)r'_{\varphi_k} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(1-\lambda_k)r'_{x_1} & \cdots & (1-\lambda_k)r'_{x_k} & (1-\lambda_k)r'_{f_1} f'_{1,y} & \cdots & \lambda_k + (1-\lambda_k)r'_{f_k} f'_{k,y} & (1-\lambda_k)r'_{\varphi_1} & \cdots & (1-\lambda_k)r'_{\varphi_k} \\
\hline
\Phi_1^{x_1} & \cdots & \Phi_1^{x_k} & 0 & \cdots & 0 & \Phi_1^{\varphi_1} & \cdots & \Phi_1^{\varphi_k} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\Phi_k^{x_1} & \cdots & \Phi_k^{x_k} & 0 & \cdots & 0 & \Phi_k^{\varphi_1} & \cdots & \Phi_k^{\varphi_k}
\end{array} \right\| \tag{D.7.7}$$

The remaining term in the characteristic polynomial of the original system is represented by the characteristic polynomial of the latter matrix, which we call $\boldsymbol{L}$. This matrix has a dimension $3k \times 3k$ when $k < N$. If $k = N$, representation (D.7.7) is, strictly speaking, not correct, since there are only

$N-1$ wealth shares φ's. In this case, the correct matrix has dimension $(3N-1)\times(3N-1)$ and can be obtained from (D.7.7) through elimination the last row and the last column. We will compute now the characteristic polynomial, i.e. determinant $\det(\boldsymbol{L}-\lambda\boldsymbol{I})$, where $\boldsymbol{I}$ denotes an identity matrix of the corresponding dimension. We consider separately the following two cases: when $k<N$ and when $k=N$.

If $k<N$, then from (D.7.2) it follows that for $n,m\le k$ it is

$$\Phi_n^{\varphi_m} = \begin{cases} 1-\varphi_n^*\,v & \text{if } n=m \\ -\varphi_n^*\,v & \text{otherwise} \end{cases}, \qquad \text{where} \qquad v = \left(x_{1\diamond k}^* - x_N^*\right)\frac{\bar{e}+r^*}{1+r^*}\,. \tag{D.7.8}$$

Moreover, since all survivors invest share $x_{1\diamond k}$, it follows from (D.7.3) that for $m\le k$

$$r'_{\varphi_m} = v\,b\,, \qquad \text{where} \qquad b = x_N^*\,\frac{1+r^*}{x_{1\diamond k}^*\left(1-x_{1\diamond k}^*\right)}\quad. \tag{D.7.9}$$

Thus, almost all elements in the last column block of (D.7.7) are proportional to the constant v. It suggests that for the computation of the characteristic polynomial of this matrix, one can represent each of these column as appropriate sum and then use the multilinear property of the determinant. In order to implement this idea, we introduce the following column vectors

$$\begin{aligned}
\boldsymbol{b} &= \left\| \begin{array}{ccc|ccc|cccc} 0 & \dots & 0 & (1-\lambda_1)\,b & \dots & (1-\lambda_k)\,b & -\varphi_1^* & -\varphi_2^* & \dots & -\varphi_k^* \end{array} \right\|, \\
\boldsymbol{b}_1 &= \left\| \begin{array}{ccc|ccc|cccc} 0 & \dots & 0 & 0 & \dots & 0 & 1-\mu & 0 & \dots & 0 \end{array} \right\|, \\
&\vdots \\
\boldsymbol{b}_k &= \left\| \begin{array}{ccc|ccc|cccc} 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1-\mu \end{array} \right\|.
\end{aligned}$$

The last column block in the determinant $\det(\boldsymbol{L}-\lambda\boldsymbol{I})$ can now be represented as

$$\left\| \, v\,\boldsymbol{b}+\boldsymbol{b}_1 \mid \dots \mid v\,\boldsymbol{b}+\boldsymbol{b}_k \, \right\| \quad,$$

and we apply the multilinear property to this block. Thus, we consider each of these columns as a sum of two terms and end up with a sum of 2^k determinants. Notice, however, that many of them are zeros, since they contain two or more columns proportional to vector $\boldsymbol{b}$. There are only $k+1$ non-zero elements in the expansion. One of them has the following structure of the last column block:

$$\left\| \, \boldsymbol{b}_1 \mid \dots \quad \dots \quad \dots \mid \boldsymbol{b}_k \, \right\| \quad,$$

while k others possess similar structure in the last column block, with column $v\,\boldsymbol{b}$ on the ν'th place instead of $\boldsymbol{b}_\nu$ for all $\nu\in\{1,\dots,k\}$:

$$\left\| \, \boldsymbol{b}_1 \mid \dots \mid v\,\boldsymbol{b} \mid \dots \mid \boldsymbol{b}_k \, \right\| \quad.$$

Matrix with the former block contains diagonal lower-right corner and, therefore, its determinant is equal to $(1-\mu)^k \det \boldsymbol{M}$, where

$$\boldsymbol{M} = \left\| \begin{array}{ccc|ccc}
-\mu & \dots & 0 & f'_{1,y} & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & -\mu & 0 & \dots & f'_{k,y} \\
\hline
(1-\lambda_1)\,r'_{x_1} & \dots & (1-\lambda_1)\,r'_{x_k} & \lambda_1-\mu+(1-\lambda_1)\,r'_{f_1}\,f'_{1,y} & \dots & (1-\lambda_1)\,r'_{f_k}\,f'_{k,y} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(1-\lambda_k)\,r'_{x_1} & \dots & (1-\lambda_k)\,r'_{x_k} & (1-\lambda_k)\,r'_{f_1}\,f'_{1,y} & \dots & \lambda_k-\mu+(1-\lambda_k)\,r'_{f_k}\,f'_{k,y}
\end{array} \right\|\,. \tag{D.7.10}$$

Other k determinants can be simplified in analogous way, so that

$$\det(\boldsymbol{L} - \lambda \boldsymbol{I}) = (1-\mu)^k \det \boldsymbol{M} + (1-\mu)^{k-1} \sum_{\nu=1}^{k} \det \boldsymbol{M}_\nu \quad , \tag{D.7.11}$$

where for all $\nu \in \{1, \ldots, k\}$ we define the following matrix

$$\boldsymbol{M}_\nu = \left\| \begin{array}{ccc|ccc|c} -\mu & \cdots & 0 & f'_{1,y} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -\mu & 0 & \cdots & f'_{k,y} & 0 \\ \hline (1-\lambda_1)\, r'_{x_1} & \cdots & (1-\lambda_1)\, r'_{x_k} & \lambda_1 - \mu + (1-\lambda_1)\, r'_{f_1}\, f'_{1,y} & \cdots & (1-\lambda_1)\, r'_{f_k}\, f'_{k,y} & (1-\lambda_1)\, v\, b \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (1-\lambda_k)\, r'_{x_1} & \cdots & (1-\lambda_k)\, r'_{x_k} & (1-\lambda_k)\, r'_{f_1}\, f'_{1,y} & \cdots & \lambda_k - \mu + (1-\lambda_k)\, r'_{f_k}\, f'_{k,y} & (1-\lambda_k)\, v\, b \\ \hline \Phi_\nu^{x_1} & \cdots & \Phi_\nu^{x_k} & 0 & \cdots & 0 & -v\, \varphi^*_\nu \end{array} \right\| \quad . \tag{D.7.12}$$

Constants v and b in the last column have been defined in (D.7.8) and (D.7.9), respectively.

Our next step will consist in the computation of the determinants of the matrices $\boldsymbol{M}$ and $\boldsymbol{M}_\nu$. We will again exploit the multilinear property of the determinant. To compute the first determinant we rewrite matrix as follows

$$\boldsymbol{M} = \left\| \begin{array}{ccc|ccc} -\mu & \cdots & 0 & f'_{1,y} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\mu & 0 & \cdots & f'_{k,y} \\ \hline \multicolumn{6}{c}{(1-\lambda_1)\, \boldsymbol{c} + \boldsymbol{c}_1} \\ \multicolumn{6}{c}{\vdots} \\ \multicolumn{6}{c}{(1-\lambda_k)\, \boldsymbol{c} + \boldsymbol{c}_k} \end{array} \right\| \quad , \tag{D.7.13}$$

where the following $k+1$ row vectors were introduced

$$\begin{aligned} \boldsymbol{c} &= \left\| \begin{array}{ccc|cccc} r'_{x_1} & \cdots & r'_{x_k} & r'_{f_1}\, f'_{1,y} & r'_{f_2}\, f'_{2,y} & \cdots & r'_{f_k}\, f'_{k,y} \end{array} \right\| \quad , \\ \boldsymbol{c}_1 &= \left\| \begin{array}{ccc|cccc} 0 & \cdots & 0 & \lambda_1 - \mu & 0 & \cdots & 0 \end{array} \right\| \quad , \\ &\vdots \\ \boldsymbol{c}_k &= \left\| \begin{array}{ccc|cccc} 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda_k - \mu \end{array} \right\| \quad , \end{aligned}$$

Applying the multilinear property to the last k rows in the determinant in (D.7.13), we get a sum of 2^k determinants. Many of them are zeros, since they contain two or more rows proportional to vector $\boldsymbol{c}$. One of the remaining determinants belongs to the lower-diagonal matrix with vectors $\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_k\}$ in the last k rows. All others non-zero determinants come from k different matrices which can be obtained from this lower-diagonal matrix by substitution of one of the rows by vector $\boldsymbol{c}$. All these determinants can be easily computed, so that we have the following

$$\begin{aligned} \det \boldsymbol{M} = (-\mu)^k \prod_{j=1}^{k} (\lambda_j - \mu) + \\ + \sum_{j=1}^{k} \left((-\mu)^{k-1} (1-\lambda_j)\, f'_{j,y} \left(-\mu\, r'_{f_j} - r'_{x_j} \right) \prod_{i=1, i \neq j}^{k} (\lambda_i - \mu) \right) \quad . \end{aligned} \tag{D.7.14}$$

Following similar procedure we can also compute the determinant of matrix $\boldsymbol{M}_\nu$. In this case for all $\nu \in \{1, \dots, k\}$ we have

$$\det \boldsymbol{M}_\nu = (-v\,\varphi^*_\nu)(-\mu)^k \prod_{j=1}^{k}(\lambda_j - \mu) + \\ + \sum_{j=1}^{k} \left((-\mu)^{k-1}\,(1-\lambda_j)\, f'_{j,y}\, v \cdot \det \boldsymbol{M}_{\nu,j} \cdot \prod_{i=1, i\neq j}^{k} (\lambda_i - \mu) \right) \quad , \tag{D.7.15}$$

where we introduced another 3×3 matrix

$$\boldsymbol{M}_{\nu,j} = \begin{Vmatrix} -\mu & 1 & 0 \\ r'_{x_j} & r'_{f_j} & b \\ \varPhi^{x_j}_\nu & 0 & -\varphi^*_\nu \end{Vmatrix} \qquad \text{with} \qquad \det \boldsymbol{M}_{\nu,j} = b\,\varPhi^{x_j}_\nu + \varphi^*_\nu \left(\mu\, r'_{f_j} + r'_{x_j}\right) \quad .$$

Now we are in the position to finish the computation of the $\det(\boldsymbol{L} - \lambda \boldsymbol{I})$ started in (D.7.11). In order to do it we have to substitute the values of the derivatives r'_{f_j} and r'_{x_j} computed in Lemma D.7.1 in the corresponding expressions. In this way, from (D.7.15) we get

$$\begin{aligned} \sum_{\nu=1}^{k} \det \boldsymbol{M}_\nu &= -v\,(-\mu)^k \prod_{j=1}^{k}(\lambda_j - \mu) + \\ &\quad + v \sum_{j=1}^{k} \left((-\mu)^{k-1}\,(1-\lambda_j)\, f'_{j,y} \prod_{i=1, i\neq j}^{k} (\lambda_i - \mu) \cdot \sum_{\nu=1}^{k} \det \boldsymbol{M}_{\nu,j} \right) = \\ &= -v\,(-\mu)^k \prod_{j=1}^{k}(\lambda_j - \mu) + \\ &\quad + v \sum_{j=1}^{k} \left((-\mu)^{k-1}\,(1-\lambda_j)\, f'_{j,y} \prod_{i=1, i\neq j}^{k} (\lambda_i - \mu) \cdot \left(\mu\, r'_{f_j} + r'_{x_j}\right) \right) = \\ &= -v\, \det \boldsymbol{M} \quad , \end{aligned}$$

where to get the second equality we observe that $\sum_{\nu=1}^{k} \varPhi^{x_j}_\nu = 0$, which can be easily verified from (D.7.2), while the last equality follows from comparison with (D.7.14). Therefore, it is

$$\det(\boldsymbol{L} - \lambda \boldsymbol{I}) = (1 - \mu - v)\,(1-\mu)^{k-1} \det \boldsymbol{M} = \\ = \left(\frac{1 + x^*_N\,(r^* + \bar{e})}{1 + r^*} - \mu \right) (1-\mu)^{k-1} \det \boldsymbol{M} \quad . \tag{D.7.16}$$

If $k = N$, i.e. all agents survive, then all investment shares are the same. In this case, according to Lemma D.7.1, all elements in the last column block of matrix (D.7.7) are zeros apart from the ones on the diagonal in the lowest $(N-1) \times (N-1)$ matrix. It contributes to the characteristic polynomial by the factor $(1-\mu)^{N-1}$. The remaining part is the determinant of matrix $\boldsymbol{M}$ in this case. This is consistent with (D.7.16).

Taking now together the contributions in (D.7.5), (D.7.6) and (D.7.16), and also computing determinant of matrix $\boldsymbol{M}$ in (D.7.14) in equilibrium, we, finally, get the characteristic polynomial (D.7.4). □

Using the characteristic polynomial of the Jacobian matrix it is straightforward to derive the equilibrium stability conditions for equilibria with $r^* \neq -\bar{e}$.

Case of one survivor: Proof of Proposition 4.4.3

If $k = 1$, characteristic polynomial (D.7.4) reduces to

$$(-1)^N \mu^{N-1} (\lambda_1 - \mu) \prod_{j=2}^{N} (\lambda_j - \mu)^2 \prod_{j=2}^{N} \left(\frac{1 + x_j^* (r^* + \bar{e})}{1 + r^*} - \mu \right)$$
$$\left(\mu(\lambda_1 - \mu) + \frac{(1 + r^*)\, \mu - 1}{x_1^*(1 - x_1^*)} \cdot (1 - \lambda_1) \cdot f'_{1,y} \right) \quad .$$

Since $0 \leq \lambda_j < 1, \forall j$ there are $3N - 2$ roots that are inside the unit circle irrespectively of model parameters. The conditions in (4.4.15) are derived from the requirement

$$\left| \frac{1 + x_j^* (r^* + \bar{e})}{1 + r^*} \right| < 1 \qquad j \geq 2 \quad ,$$

which comes from the factor before the last. The three inequalities in (4.4.14) are obtained in standard way, applying for the roots of the second degree polynomial in the last parentheses Proposition A.0.1 from Appendix A together with relation (D.2.2).

Case of many survivors: Proof of Proposition 4.4.4

In the case of $k > 1$ survivors the characteristic polynomial in (D.7.4) possesses a unit root with multiplicity $k - 1$. Consequently, the fixed point is non-hyperbolic.

Let us find the eigenspace associated to eigenvalue 1. We subtract from initial Jacobian matrix (D.7.1) computed at the equilibrium the identity matrix of the corresponding dimension and analyze the kernel of the resulting matrix $\boldsymbol{J} - \boldsymbol{I}$. This can be done through the analysis of the kernel of matrix obtained by the substitution of the identity matrix from (D.7.7). Let us consider the $k < N$ and the $k = N$ cases separately.

When $k < N$, as we showed in the proof of Lemma D.7.3, in the resulting matrix the last $k - 1$ columns are identical, see (D.7.8) and (D.7.9). Therefore, the kernel of the matrix $\boldsymbol{J} - \boldsymbol{I}$ can be generated by a basis containing the following $k - 1$ vectors

$$\boldsymbol{u}_n = \Big(\underbrace{0, \ldots, 0}_{N}; \underbrace{0, \ldots, 0}_{N}; \underbrace{0, \ldots, 0}_{N}; \underbrace{0, \ldots, 0}_{n-1}, 1, \underbrace{0, \ldots, 0}_{k-n-1}, -1; \underbrace{0, \ldots, 0}_{N-1-k} \Big) , \quad 1 \leq n \leq k - 1 \quad . \tag{D.7.17}$$

Notice that the direction of vector $\boldsymbol{u}_n$ corresponds to a change in the relative wealths of the n-th and k-th survivor.

If, instead, $k = N$, then the last $k - 1$ columns in the resulting matrix are zero vectors, and then the kernel of the matrix $\boldsymbol{J} - \boldsymbol{I}$ can be generated with the $N - 1$ vectors of the canonical basis

$$\boldsymbol{v}_n = \Big(\underbrace{0, \ldots, 0}_{N}; \underbrace{0, \ldots, 0}_{N}; \underbrace{0, \ldots, 0}_{N}; \underbrace{0, \ldots, 0}_{n-1}, 1, \underbrace{0, \ldots, 0}_{N-n-1} \Big) , \quad 1 \leq n \leq N - 1 \quad . \tag{D.7.18}$$

whose direction corresponds to a change in the relative wealths of the n-th and N-th survivors.

If the system is perturbed away from equilibrium $\boldsymbol{x}^*$ along the directions defined in (D.7.17) or (D.7.18), a new fixed point is reached. Then, the system is stable, but not asymptotically stable, with respect to these perturbations.

Moreover, since the eigenspaces identified above do not depend on the system parameters, it is immediate to realize that they do constitute not only the tangent spaces to the corresponding non-hyperbolic manifolds, but the manifolds themselves.

The polynomial (4.4.16) is the last factor in (D.7.4), while conditions (4.4.17) are obtained by imposing

$$\left| \frac{1 + x_j^* (r^* + \bar{e})}{1 + r^*} \right| < 1 \qquad j > k + 1 \,,$$

which completes the proof.

Case of many survivors with same λ: Proof of Corollary 4.4.3

If all the survivors are characterized by the same parameter $\lambda \in [0,1)$, the last factor in (D.7.4) reduces to

$$(\lambda-\mu)^{k-1}\left(\mu\,(\lambda-\mu)+(1-\lambda)\,\frac{(1+r^*)\mu-1}{x^*_{1\diamond k}(1-x^*_{1\diamond k})}\sum_{j=1}^{k}\varphi^*_j\, f'_{j,y}\right)$$

and the result directly follows applying Proposition A.0.1 to the second-degree polynomial in the parenthesis above.

D.7.2 Equilibria with $r^* = -\bar{e}$.

Let us now move to those equilibria which we found in Proposition 4.4.2 and suppose that $k \leq N$ agents survive in this equilibrium. As usually, the Jacobian structure in the fixed point can be established by the evaluation of derivatives of corresponding functions. We provide these derivatives in the following

Lemma D.7.4. *Consider equilibrium $\boldsymbol{x}^*$ of system* (4.4.1) *with k survivors and $r^* = -\bar{e}$. In this equilibrium functions Φ_n defined in* (4.4.2) *for all $n \in \{1,\ldots,N-1\}$ have the following derivatives:*

$$\begin{aligned}
&\Phi_n^{x_m} = 0 && \forall m \in \{1,\ldots,N\} \quad ,\\
&\Phi_n^{\varphi_m} = \delta_{n,m} && \forall m \in \{1,\ldots,N-1\} \quad ,\\
&\Phi_n^{r} \;= \varphi^*_n\, x^*_n \quad ,
\end{aligned} \tag{D.7.19}$$

where $\delta_{n,m}$ is the Kronecker delta. Furthermore, the expression in the right-hand side of (4.4.3)*, providing the price return, has the following derivatives in this equilibrium*

$$\begin{aligned}
&r'_{x_m} = \varphi^*_m\,\frac{1-\bar{e}}{\langle x^2\rangle} && \forall m \in \{1,\ldots,N\} \quad ,\\
&r'_{f_m} = -\varphi^*_m\,\frac{1}{\langle x^2\rangle} && \forall m \in \{1,\ldots,N\} \quad ,\\
&r'_{\varphi_m} = -\bar{e}\,\frac{x^*_m - x^*_N}{\langle x^2\rangle} && \forall m \in \{1,\ldots,N-1\} \quad .
\end{aligned} \tag{D.7.20}$$

where $\langle x^2\rangle$ denotes the weighted sum of squares of the equilibrium investment shares:

$$\langle x^2\rangle = \sum_{j=1}^{k}\varphi^*_j {x^*_j}^2 \quad . \tag{D.7.21}$$

Analogously to Lemma D.7.1, this statement can be proven by the direct differentiation. In the process of computations we use the relation $r^* = -\bar{e}$ which characterizes given equilibrium, as well as equality (4.4.19).

With results of Lemma D.7.4 one can simplify the Jacobian (D.7.1). Indeed, for all agents (with corresponding indeces) who do not survive in the equilibrium $\Phi_n^r = 0$, $r'_{x_m} = 0$ and $r'_{f_m} = 0$. It implies

$$\left[\frac{\partial \mathcal{W}}{\partial \mathcal{X}}\right]_{n,m} = \begin{cases} \Phi^r_n \, r'_{x_m} & m,n \le k \\ 0 & \text{otherwise} \end{cases}, \qquad \left[\frac{\partial \mathcal{W}}{\partial \mathcal{Y}}\right]_{n,m} = \begin{cases} \Phi^r_n \, r'_{f_m} \, f'_{m,y} & m,n \le k \\ 0 & \text{otherwise} \end{cases}$$

$$\left[\frac{\partial \mathcal{W}}{\partial \mathcal{Z}}\right]_{n,m} = \begin{cases} \Phi^r_n \, r'_{f_m} \, f'_{m,z} & m,n \le k \\ 0 & \text{otherwise} \end{cases}, \qquad \left[\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\right]_{n,m} = \begin{cases} \delta_{n,m} + \Phi^r_n \, r'_{\varphi_m} & n \le k \\ \delta_{n,m} & \text{otherwise} \end{cases}$$

These results lead to some simplifications in the last row of the blocks of the Jacobian. Applying corresponding simplifications to the second row of the blocks whenever it is possible, one get the structure of the Jacobian in equilibrium which we present in the following statement which is analogous to Lemma D.7.2.

Lemma D.7.5. *Let $\boldsymbol{x}^*$ be an equilibrium of system* (4.4.1) *with k survivors and $r^* = -\bar{e}$. The Jacobian matrix computed in this point $\boldsymbol{J}(\boldsymbol{x}^*)$ has the structure of the following matrix, where each one of the* 16 *blocks defined above is divided into* 4 *sub-blocks to display the upper-left $k \times k$ minor matrix. All non-zero elements are denoted by the same symbol "$\star$".*

$$\left\| \begin{array}{cccccc|cccccc|cccccc|cccccc}
0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \star & 0 & \dots & 0 & 0 & \dots & \star & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \star & 0 & \dots & 0 & 0 & \dots & \star & 0 & \dots & 0 & 0 & \dots & 0 \\
\hline
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star \\
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & 0 & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & 0 & \dots & \star & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star \\
\hline
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \star & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \star & 0 & \dots & 0 & 0 & \dots & 0 \\
\hline
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \star
\end{array} \right\|$$

Lemma D.7.6. *The characteristic polynomial P_J of the matrix $\boldsymbol{J}(\boldsymbol{x}^*)$ in the equilibrium with $r^* =$*

$-\bar{e}$ *reads*

$$P_J(\mu) = (-1)^{N-1}\, \mu^{N-1}\, (1-\mu)^{N-2}\, (\mu - 1 + \bar{e}) \prod_{j=1}^{k} (\lambda_j - \mu) \prod_{j=k+1}^{N} (\lambda_j - \mu)^2$$
$$\left(\mu \prod_{j=1}^{k} (\lambda_j - \mu) + \frac{1-\mu}{\langle x^2 \rangle} \sum_{j=1}^{k} \Big(\varphi_j^*\, f'_{j,y} (1 - \lambda_j) \prod_{i=1, i\neq j}^{k} (\lambda_i - \mu) \Big) \right) \tag{D.7.22}$$

Proof. The proof is constructive and analogous to the proof of Lemma D.7.3. Moreover, first steps of that proof we can repeat here. Indeed, in the Jacobian matrix in Lemma D.7.5 it is: (i) each of the N rows belonging to the third row of blocks the only non-zero entries are the λ's on the diagonal of $[\partial \mathcal{Z}/\partial \mathcal{Z}]$, (ii) in each of the last $N-1-k$ rows of the matrix, the only non-zero entries belong to the main diagonal of $[\partial \mathcal{W}/\partial \mathcal{W}]$ and equal to 1, (iii) the last $N-k$ columns of the leftmost blocks contain only zero entries. Together, it gives the first entry in the characteristic polynomial:

$$(-\mu)^{N-k}\, (1-\mu)^{N-1-k} \prod_{j=1}^{N} \big(\lambda_j - \mu\big) \quad . \tag{D.7.23}$$

(If all N agents survive in equilibrium the second term in the expression above is absent.) In order to find the remaining factors we eliminate the rows and columns associated to the previous eigenvalues. Consider now the last $N-k$ columns in the second blocks column in the remaining matrix. The only non zero elements are on the main diagonal of $[\partial \mathcal{Y}/\partial \mathcal{Y}]$. This leads, if $k < N$, to a second contribution

$$\prod_{j=k+1}^{N} \big(\lambda_j - \mu\big) \quad . \tag{D.7.24}$$

After the corresponding further elimination of rows and columns the following matrix is obtained

$$\left\| \begin{array}{ccc|ccc|ccc} 0 & \cdots & 0 & f'_{1,y} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & f'_{k,y} & 0 & \cdots & 0 \\ \hline (1-\lambda_1) r'_{x_1} & \cdots & (1-\lambda_1) r'_{x_k} & \lambda_1 + (1-\lambda_1) r'_{f_1} f'_{1,y} & \cdots & (1-\lambda_1) r'_{f_k} f'_{k,y} & (1-\lambda_1) r'_{\varphi_1} & \cdots & (1-\lambda_1) r'_{\varphi_k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (1-\lambda_k) r'_{x_1} & \cdots & (1-\lambda_k) r'_{x_k} & (1-\lambda_k) r'_{f_1} f'_{1,y} & \cdots & \lambda_k + (1-\lambda_k) r'_{f_k} f'_{k,y} & (1-\lambda_k) r'_{\varphi_1} & \cdots & (1-\lambda_k) r'_{\varphi_k} \\ \hline \Phi^r_1 r'_{x_1} & \cdots & \Phi^r_1 r'_{x_k} & \Phi^r_1 r'_{f_1} f'_{1,y} & \cdots & \Phi^r_1 r'_{f_k} f'_{k,y} & 1 + \Phi^r_1 r'_{\varphi_1} & \cdots & \Phi^r_1 r'_{\varphi_k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Phi^r_k r'_{x_1} & \cdots & \Phi^r_k r'_{x_k} & \Phi^r_k r'_{f_1} f'_{1,y} & \cdots & \Phi^r_k r'_{f_k} f'_{k,y} & \Phi^r_k r'_{\varphi_1} & \cdots & 1 + \Phi^r_k r'_{\varphi_k} \end{array} \right\| \tag{D.7.25}$$

Notice that this matrix, as it is written, has dimension $3k \times 3k$. Strictly speaking, this is not always correct. The problematic case is one with $k = N$, when the correct matrix has dimension $(3N-1) \times (3N-1)$ due to the fact that there are only $N-1$ wealth shares. Thus, when $k = N$ the lower row block and the rightmost column block have, correspondingly, $N-1$ rows and $N-1$ columns. In our further computations there are no big differences between cases $k < N$ and $k = N$. These computations are, formally, valid for the case $k < N$. Corresponding differences for $k = N$ we mention in the footnotes. We denote matrix (D.7.25), shortened if $k = N$, as $\boldsymbol{L}$ and compute the remaining term in the characteristic polynomial of the original system as $\det(\boldsymbol{L} - \lambda \boldsymbol{I})$, where $\boldsymbol{I}$ is an identity matrix of the corresponding dimension.

We apply the multilinear property of the determinant to the last column block in matrix (D.7.25).

In order to implement this idea, we introduce the following column vectors of the length[1] $3k$:

$$\begin{aligned}
\boldsymbol{d} &= \left\| \begin{array}{ccc|ccc|cccc} 0 & \dots & 0 & 1-\lambda_1 & \dots & 1-\lambda_k & \Phi^r_1 & \Phi^r_2 & \dots & \Phi^r_k \end{array} \right\| , \\
\boldsymbol{d}_1 &= \left\| \begin{array}{ccc|ccc|cccc} 0 & \dots & 0 & 0 & \dots & 0 & 1-\mu & 0 & \dots & 0 \end{array} \right\| , \\
&\vdots \\
\boldsymbol{d}_k &= \left\| \begin{array}{ccc|ccc|cccc} 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1-\mu \end{array} \right\| .
\end{aligned}$$

The last column block in the determinant $\det(\boldsymbol{L}-\lambda\boldsymbol{I})$ can now be represented as

$$\left\| \, r'_{\varphi_1}\,\boldsymbol{d}+\boldsymbol{d}_1 \mid \dots \mid r'_{\varphi_k}\,\boldsymbol{d}+\boldsymbol{d}_k \, \right\| ,$$

and we apply the multilinear property to this block. Thus, we consider each of these columns as a sum of two terms and end up with a sum of 2^k determinants. Notice, however, that many of them are zeros, since they contain two or more columns proportional to vector $\boldsymbol{d}$. There are only $k+1$ non-zero elements in the expansion[2]. One of them has the following structure of the last column block:

$$\left\| \, \boldsymbol{d}_1 \mid \dots \quad \dots \quad \dots \mid \boldsymbol{d}_k \, \right\| ,$$

while k others possess similar structure in the last column block, with column $r'_{\varphi_\nu}\,\boldsymbol{d}$ on the ν'th place instead of $\boldsymbol{d}_\nu$ for all $\nu \in \{1,\dots,k\}$:

$$\left\| \, \boldsymbol{d}_1 \mid \dots \mid r'_{\varphi_\nu}\,\boldsymbol{d} \mid \dots \mid \boldsymbol{d}_k \, \right\| .$$

Matrix with the former block contains the diagonal lower-right corner and, therefore, its determinant is equal to $(1-\mu)^k \det \boldsymbol{N}$, where matrix $\boldsymbol{N}$ is identical to the matrix $\boldsymbol{M}$ defined in (D.7.10). We use here another notation, however, in order to stress that the partial derivatives r'_{x_j} and r'_{f_j} used in these two matrices have different values in different equilibria. For the equilibrium which we consider here, these derivatives were computed in Lemma D.7.4.

Analogously, other k determinants can be represented as $(1-\mu)^{k-1} \det \boldsymbol{N}_\nu$, where for all $\nu \in \{1,\dots,k\}$ we define matrix

$$\boldsymbol{N}_\nu = \left\| \begin{array}{ccc|ccc|c}
-\mu & \dots & 0 & f'_{1,y} & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & -\mu & 0 & \dots & f'_{k,y} & 0 \\
\hline
(1-\lambda_1)\,r'_{x_1} & \dots & (1-\lambda_1)\,r'_{x_k} & \lambda_1-\mu+(1-\lambda_1)\,r'_{f_1}\,f'_{1,y} & \dots & (1-\lambda_1)\,r'_{f_k}\,f'_{k,y} & (1-\lambda_1)\,r'_{\varphi_\nu} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(1-\lambda_k)\,r'_{x_1} & \dots & (1-\lambda_k)\,r'_{x_k} & (1-\lambda_k)\,r'_{f_1}\,f'_{1,y} & \dots & \lambda_k-\mu+(1-\lambda_k)\,r'_{f_k}\,f'_{k,y} & (1-\lambda_k)\,r'_{\varphi_\nu} \\
\hline
\Phi^r_\nu r'_{x_1} & \dots & \Phi^r_\nu r'_{x_k} & \Phi^r_\nu r'_{f_1} f'_{1,y} & \dots & \Phi^r_\nu r'_{f_k} f'_{k,y} & \Phi^r_\nu r'_{\varphi_\nu}
\end{array} \right\| . \qquad \text{(D.7.26)}$$

To compute the determinant of this matrix we again use the multilinear property of the determinant, applying it to the row vectors in the second block (among three horizontal blocks separated by the straight lines). For all $\nu \in \{1,\dots,k\}$ we introduce vectors

$$\begin{aligned}
\boldsymbol{e}^\nu &= \left\| \begin{array}{ccc|cccc|c} r'_{x_1} & \dots & r'_{x_k} & r'_{f_1}\,f'_{1,y} & r'_{f_2}\,f'_{2,y} & \dots & r'_{f_k}\,f'_{k,y} & r'_{\varphi_\nu} \end{array} \right\| , \\
\boldsymbol{e}^\nu_1 &= \left\| \begin{array}{ccc|cccc|c} 0 & \dots & 0 & \lambda_1-\mu & 0 & \dots & 0 & 0 \end{array} \right\| , \\
&\vdots \\
\boldsymbol{e}^\nu_k &= \left\| \begin{array}{ccc|cccc|c} 0 & \dots & 0 & 0 & 0 & \dots & \lambda_k-\mu & 0 \end{array} \right\| ,
\end{aligned}$$

[1] Of course, when $k=N$ agents survive, these vectors have length $3N-1$ each. Moreover, in this case there is vector $\boldsymbol{d}$ and only $N-1$ vectors $\boldsymbol{d}_1 \dots \boldsymbol{d}_{N-1}$.

[2] Correspondingly, if $k=N$ there are only N elements in the expansion. In particular, index ν introduced below belongs to the set $\{1,\dots,N-1\}$ in this case.

so that the second block in (D.7.26) can be represented as

$$\left\| \begin{array}{c} (1-\lambda_1)\,\boldsymbol{e}^\nu + \boldsymbol{e}_1^\nu \\ \vdots \\ (1-\lambda_k)\,\boldsymbol{e}^\nu + \boldsymbol{e}_k^\nu \end{array} \right\| \quad ,$$

Applying now the multilinear property to these k rows, we get a sum of 2^k determinants. Many of them are zeros, since they contain two or more rows proportional to vector $\boldsymbol{e}^\nu$. One of the remaining determinants contains in the middle part the rows $\{\boldsymbol{e}_1^\nu, \ldots, \boldsymbol{e}_k^\nu\}$, so that corresponding matrix has only diagonal block in the middle part. All other non-zero determinants come from k different matrices obtained from this matrix by substitution of one of the rows by vector $\boldsymbol{e}^\nu$. All these determinants can be easily computed, so that we have the following

$$\begin{aligned} \det \boldsymbol{N}_\nu = {} & \Phi_\nu^r \, r'_{\varphi_\nu} \, (-\mu)^k \prod_{j=1}^{k} (\lambda_j - \mu) + \\ & + \sum_{j=1}^{k} \left(\Phi_\nu^r \, r'_{\varphi_\nu} \, (-\mu)^{k-1} \, (1-\lambda_j) \, f'_{j,y} \cdot \det \boldsymbol{N}_{\nu,j} \cdot \prod_{i=1, i\neq j}^{k} (\lambda_i - \mu) \right) \quad , \end{aligned} \tag{D.7.27}$$

where we introduced another 3×3 matrix

$$\boldsymbol{N}_{\nu,j} = \left\| \begin{array}{ccc} -\mu & 1 & 0 \\ r'_{x_j} & r'_{f_j} & 1 \\ r'_{x_j} & r'_{f_j} & 1 \end{array} \right\| \quad . \tag{D.7.28}$$

However all matrices $\boldsymbol{N}_{\nu,j}$ possess two identical rows and, therefore, have a zero determinant, so that $\det \boldsymbol{N}_\nu$ is provided by only the first term in (D.7.27). Using the expressions for Φ_ν^r and r'_{φ_ν} from (D.7.19) and (D.7.20), one can directly check[3] that $\sum_{\nu=1}^{k} \Phi_\nu^r r'_{\varphi_\nu} = -\bar{e}$ and also compute $\det \boldsymbol{N}$ using formula (D.7.14) in our equilibrium. Finally, we have:

$$\begin{aligned} \det(\boldsymbol{L} - \lambda \boldsymbol{I}) = {} & (1-\mu)^k \det \boldsymbol{N} + (1-\mu)^{k-1} \sum_{\nu=1}^{k} \det \boldsymbol{N}_\nu = \\ & = (1-\mu)^{k-1} \, (-\mu)^{k-1} \Bigg((1-\mu)(-\mu) \prod_{j=1}^{k} (\lambda_j - \mu) + \\ & \qquad + (1-\mu) \, \frac{\mu - 1 + \bar{e}}{\langle x^2 \rangle} \sum_{j=1}^{k} \Big((1-\lambda_j) \, f'_{j,y} \, \varphi_j^* \prod_{i=1, i\neq j}^{k} (\lambda_i - \mu) \Big) + \\ & \qquad + (-\mu)(-\bar{e}) \prod_{j=1}^{k} (\lambda_j - \mu) \Bigg) = \\ & = (1-\mu)^{k-1} \, (-\mu)^{k-1} (\mu - 1 + \bar{e}) \\ & \qquad \left(\mu \prod_{j=1}^{k} (\lambda_j - \mu) + \frac{1-\mu}{\langle x^2 \rangle} \sum_{j=1}^{k} \Big((1-\lambda_j) \, f'_{j,y} \, \varphi_j^* \prod_{i=1, i\neq j}^{k} (\lambda_i - \mu) \Big) \right) \end{aligned} \tag{D.7.29}$$

Combining now (D.7.23), (D.7.24) and (D.7.29) we get the polynomial (D.7.22). □

[3] This is the place where the difference between the case with $k < N$ and $k = N$ matters. The point is that in the latter situation there are only $N-1$ matrices $\boldsymbol{N}_\nu$. However, the result of the summation is the same in both cases.

Using the characteristic polynomial of the Jacobian matrix it is straightforward to derive the equilibrium stability conditions for equilibria with $r^* = -\bar{e}$.

Proof of Proposition 4.4.5

Independently of the number of survivors, the characteristic polynomial in (D.7.22) possesses a unit root with multiplicity $N-2$. Consequently, the fixed point $\boldsymbol{x^*}$ is never hyperbolic, when $N \geq 3$.

In Corollary 4.4.2 we have found that there is a manifold of equilibria of dimension $N-2$. Moving from $\boldsymbol{x^*}$ along this manifold we reach new fixed points. Therefore, this manifold represents non-hyperbolic manifold of the fixed point $\boldsymbol{x^*}$. For the stability of equilibrium $\boldsymbol{x^*}$ with respect to the perturbations in the directions *orthogonal* to this manifold, it is sufficient to have all other eigenvalues inside the unit circle. If this condition is satisfied, then equilibrium $\boldsymbol{x^*}$ of the system is stable, but not asymptotically stable.

Since $\lambda_n < 1$ for all n and since $\bar{e} > 0$, this sufficient condition can be expressed through the roots of the last term in (D.7.22). This term is exactly polynomial (4.4.20).

Case of many survivors with same λ: Proof of Corollary 4.4.3

If all the survivors are characterized by the same parameter $\lambda \in [0, 1)$, the last factor in (D.7.22) reduces to

$$(\lambda - \mu)^{k-1} \left(\mu \, (\lambda - \mu) + (1 - \lambda) \, \frac{1-\mu}{\left\langle x^2 \right\rangle} \sum_{j=1}^{k} \varphi_j^* \, f'_{j,y} \right)$$

and the result directly follows from Proposition A.0.1.

Appendix E

Proofs of Propositions in Chapter 5

E.1 Determinant of auxiliary matrix

The following result is useful for the stability analysis of the different systems considered in Chapter 5.

Lemma E.1.1.

$$\begin{vmatrix} x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n \\ 1 & -\mu & 0 & \dots & 0 & 0 \\ 0 & 1 & -\mu & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\mu & 0 \\ 0 & 0 & 0 & \dots & 1 & -\mu \end{vmatrix} = (-1)^{n-1} \sum_{k=1}^{n} x_k \, \mu^{n-k} \quad , \tag{E.1.1}$$

Proof. Consider this determinant as a sum of elements from the first row multiplied on the corresponding minor. The minor of element x_k is a block-diagonal matrix consisting of two blocks. The upper-left block is an upper-diagonal matrix with 1's on the diagonal. The lower-right block is a lower-diagonal matrix with $-\mu$'s on the diagonal. The determinant of this minor is equal to $(-\mu)^{n-1-k}$ and the relation above immediately follows. □

E.2 Proof of Proposition 5.2.1

In order to get the first equality in (5.2.4) one has to plug the equilibrium values of the variables in the first equation in (5.2.2). From the second equation of (5.2.2) one has

$$r^* = R(x^*, x^*, \bar{e}) = \bar{e} \, \frac{x^*}{1 - x^*} \quad .$$

Inverting this relation to obtain x^* as a function of r^* and using the first equality in (5.2.4) one get the second equality. Item *(ii)* follows directly from condition (3.6.3) written at equilibrium. Item *(iii)* follows from (3.6.5) and the results in item *(i)*:

$$\rho^* = x^* \, (r^* + \bar{e}) = l(r^*) \, (r^* + \bar{e}) = r^* \quad .$$

E.3 Proof of Proposition 5.2.2

The condition for the stability is a direct consequence of the characteristic polynomial of the Jacobian matrix at equilibrium. The $(L+1) \times (L+1)$ Jacobian matrix $\boldsymbol{J}$ of system (5.2.2) reads

$$\begin{Vmatrix} 0 & \frac{\partial f}{\partial r_0} & \frac{\partial f}{\partial r_1} & \frac{\partial f}{\partial r_2} & \dots & \frac{\partial f}{\partial r_{L-2}} & \frac{\partial f}{\partial r_{L-1}} \\ R^x & R^f \frac{\partial f}{\partial r_0} & R^f \frac{\partial f}{\partial r_1} & R^f \frac{\partial f}{\partial r_2} & \dots & R^f \frac{\partial f}{\partial r_{L-2}} & R^f \frac{\partial f}{\partial r_{L-1}} \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{Vmatrix} , \tag{E.3.1}$$

where

$$R^x = \frac{\partial R(x^*, x^*)}{\partial x} = -\frac{1}{x^*\,(1-x^*)} \,, \qquad R^f = \frac{\partial R(x^*, x^*)}{\partial x'} = \frac{1+r^*}{x^*\,(1-x^*)} \,. \tag{E.3.2}$$

The stability condition of equilibrium are provided by the following

Lemma E.3.1. *The characteristic polynomial $P_J(\mu)$ of system* (5.2.2) *in the equilibrium $\boldsymbol{x}^*$ is*

$$P_J(\mu) = (-1)^{L-1} \left(\mu^{L+1} - \frac{(1+r^*)\mu - 1}{x^*(1-x^*)} P_f(\mu) \right) \tag{E.3.3}$$

where $P_f(\mu)$ denotes the stability polynomial of function f introduced in (5.2.5).

Proof. Consider (E.3.1) and introduce $(L+1) \times (L+1)$ identity matrix $\boldsymbol{I}$. Expanding the determinant of $\boldsymbol{J} - \mu\,\boldsymbol{I}$ by the elements of the first column and using result of Lemma E.1.1 from Appendix E.1 one has

$$\begin{aligned} \det\big(\boldsymbol{J} - \mu\,\boldsymbol{I}\big) &= (-\mu)\,(-1)^{L-1} \left(\left(R^f \frac{\partial f}{\partial r_0} - \mu \right) \mu^{L-1} + R^f \frac{\partial f}{\partial r_1}\,\mu^{L-2} + \cdots + R^f \frac{\partial f}{\partial r_{L-1}} \right) - \\ &\quad - R^x\,(-1)^{L-1} \left(\frac{\partial f}{\partial r_0}\,\mu^{L-1} + \frac{\partial f}{\partial r_1}\,\mu^{L-2} + \cdots + \frac{\partial f}{\partial r_{L-2}}\,\mu + \frac{\partial f}{\partial r_{L-1}} \right) = \\ &= (-1)^{L-1} \left(\mu^{L+1} - \left(\mu R^f + R^x \right) \sum_{k=0}^{L-1} \frac{\partial f}{\partial r_k}\,\mu^{L-1-k} \right) \end{aligned} ,$$

which, using relations in (E.3.2) and definition of stability polynomial in (5.2.5) reduces to (E.3.3). ☐

Using the relationship $l'(r^*) = x^*(1-x^*)/r^*$ it is immediate to see that, apart from irrelevant sign, (E.3.3) is identical to (5.2.7).

E.4 Proof of Proposition 5.3.1

From block $\mathcal{X}$ one immediately has (5.3.8) and (5.3.12). From block $\mathcal{W}$ using (5.3.5) and the condition $r^* + \bar{e} \neq 0$ one obtains

$$\varphi_n^* = 0 \quad \text{or} \quad \sum\nolimits_{m=1}^{N-1} \varphi_m^*\,x_m^* + \left(1 - \sum\nolimits_{m=1}^{N-1} \varphi_m^* \right) x_N^* = x_n^* \quad \forall n \in \{1, \dots, N-1\} \quad . \tag{E.4.1}$$

Finally, from the first row of block $\mathcal{R}$ it is

$$r^* = \bar{e}\,\frac{\sum_{n=1}^{N-1}\varphi_n^* {x_n^*}^2 + \left(1-\sum_{n=1}^{N-1}\varphi_n^*\right){x_N^*}^2}{\sum_{n=1}^{N-1}\varphi_n^*\, x_n^*\,(1-x_n^*) + \left(1-\sum_{n=1}^{N-1}\varphi_n^*\right)x_N^*\,(1-x_N^*)}\,. \tag{E.4.2}$$

The previous set of equations admits two types of solutions, depending on how many equilibrium wealth shares are different from zero: if one or many.

To derive the first type of solutions assume (5.3.6). In this case (E.4.1) is satisfied for all agents. From (E.4.2) one has $x_1^* = r^*/(\bar{e}+r^*)$ which together with (5.3.8) leads to (5.3.7).

To derive the second type of solutions assume (5.3.9). In this case, the second equality of (E.4.1) must be satisfied for any $n \le k$. Since its left-hand side does not depend on n, a $x_{1\diamond k}^*$ must exist such that $x_1^* = \cdots = x_k^* = x_{1\diamond k}^*$. Substituting $x_n^* = 0$ for $n > k$ and $x_n^* = x_{1\diamond k}^*$ for $n \le k$ in (E.4.2) one gets $x_{1\diamond k}^* = r^*/(\bar{e}+r^*)$. The equilibrium return r^* is implicitly defined combining this last relation with (5.3.12) for $n \le k$.

The equilibrium wealth growth rate of the survivors is immediately obtained from (3.6.5) and from (5.3.8) or (5.3.12) for the single survivor and the many survivors case, respectively.

E.5 Proofs of Propositions in Section 5.3.3

Before proving Propositions 5.3.3, 5.3.4 and 5.3.5 we need some preliminary results. The Jacobian matrix of the deterministic skeleton of system (5.3.3) is a $\left(2N+L-1\right)\times\left(2N+L-1\right)$ matrix. Using the block structure introduced in Section 5.3.1 it is separated in nine blocks

$$\boldsymbol{J} = \begin{Vmatrix} \frac{\partial\mathcal{X}}{\partial\mathcal{X}} & \frac{\partial\mathcal{X}}{\partial\mathcal{W}} & \frac{\partial\mathcal{X}}{\partial\mathcal{R}} \\ \frac{\partial\mathcal{W}}{\partial\mathcal{X}} & \frac{\partial\mathcal{W}}{\partial\mathcal{W}} & \frac{\partial\mathcal{W}}{\partial\mathcal{R}} \\ \frac{\partial\mathcal{R}}{\partial\mathcal{X}} & \frac{\partial\mathcal{R}}{\partial\mathcal{W}} & \frac{\partial\mathcal{R}}{\partial\mathcal{R}} \end{Vmatrix}\,, \tag{E.5.1}$$

The block $\partial\mathcal{X}/\partial\mathcal{X}$ is a $N\times N$ matrix containing the partial derivatives of the agents' present investment choices with respect to the agents' past investment choices. According to (5.3.1) the investment choice of any agent does not explicitly depend on the investment choices in previous period and it is

$$\left[\frac{\partial\mathcal{X}}{\partial\mathcal{X}}\right]_{n,m} = \frac{\partial f_n}{\partial x_m} = 0\,, \qquad 1 \le n, m \le N$$

and this block is a zero matrix.

The block $\partial\mathcal{X}/\partial\mathcal{W}$ is a $N\times(N-1)$ matrix containing the partial derivatives of the agents' investment choices with respect to the agents' wealth shares. According to (5.3.1) this is a zero matrix and

$$\left[\frac{\partial\mathcal{X}}{\partial\mathcal{W}}\right]_{n,m} = \frac{\partial f_n}{\partial\varphi_m} = 0\,, \qquad 1 \le n \le N\,, \quad 1 \le m \le N-1\,.$$

The block $\partial\mathcal{X}/\partial\mathcal{R}$ is a $N\times L$ matrix containing the partial derivatives of the agents' investment choices with respect to the past returns

$$\left[\frac{\partial\mathcal{X}}{\partial\mathcal{R}}\right]_{n,l} = \frac{\partial f_n}{\partial r_{l-1}} = f_n^{r_{l-1}}\,, \qquad 1 \le n \le N\,, \quad 1 \le l \le L$$

The block $\partial\mathcal{W}/\partial\mathcal{X}$ is $(N-1)\times N$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' investment choices. It is

$$\left[\frac{\partial\mathcal{W}}{\partial\mathcal{X}}\right]_{n,m} = \frac{\partial\varphi_n}{\partial x_m} = \Phi_n^{x_m} + \Phi_n^R\cdot R^{x_m}\,, \qquad 1 \le n \le N-1\,, \quad 1 \le m \le N \tag{E.5.2}$$

where $R^{x_m} = \partial R/\partial x_m$, $\Phi_n^{x_m} = \partial \Phi_n/\partial x_m$ and $\Phi_n^R = \partial \Phi_n/\partial R$.

The block $\partial \mathcal{W}/\partial \mathcal{W}$ is a $(N-1)\times(N-1)$ matrix containing the partial derivatives of the agents' wealth shares with respect to the agents' wealth shares. It is

$$\left[\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\right]_{n,m} = \frac{\partial \varphi_n}{\partial \varphi_m} = \Phi_n^{\varphi_m} + \Phi_n^R \cdot R^{\varphi_m} \ , \qquad 1 \le n, m \le N-1 \ , \tag{E.5.3}$$

where $\Phi_n^{\varphi_m} = \partial \Phi_n/\partial \varphi_m$, $R^{\varphi_m} = \partial R/\partial \varphi_m$.

The block $\partial \mathcal{W}/\partial \mathcal{R}$ is a $(N-1)\times L$ matrix containing the partial derivatives of the agents' wealth share with respect to lagged returns. It is

$$\left[\frac{\partial \mathcal{W}}{\partial \mathcal{R}}\right]_{n,l} = \frac{\partial \varphi_n}{\partial r_{l-1}} = \Phi_n^R \cdot \sum_{m=1}^{N} R^{f_m} f_m^{r_{l-1}} \ , \quad 1 \le n \le N-1 \ , \quad 1 \le l \le L \ , \tag{E.5.4}$$

where $R^{f_n} = \partial R/\partial y_n$.

The block $\partial \mathcal{R}/\partial \mathcal{X}$ is the $L \times N$ matrix containing the partial derivatives of the lagged returns with respect to the agents' investment choices. Its structure is simple and reads

$$\left[\frac{\partial \mathcal{R}}{\partial \mathcal{X}}\right] = \begin{Vmatrix} R^{x_1} & R^{x_2} & \dots & R^{x_N} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{Vmatrix} \quad .$$

The block $\partial \mathcal{R}/\partial \mathcal{W}$ is the $L \times (N-1)$ matrix containing the partial derivatives of the lagged returns with respect to the agents' wealth shares and reads

$$\left[\frac{\partial \mathcal{R}}{\partial \mathcal{W}}\right] = \begin{Vmatrix} R^{\varphi_1} & R^{\varphi_2} & \dots & R^{\varphi_{N-1}} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{Vmatrix} \quad .$$

The block $\partial \mathcal{R}/\partial \mathcal{R}$ is the $L \times L$ matrix containing the partial derivatives of the lagged returns with respect to themselves

$$\left[\frac{\partial \mathcal{R}}{\partial \mathcal{R}}\right] = \begin{Vmatrix} R^{r_0} & R^{r_1} & \dots & R^{r_{L-1}} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{Vmatrix} \quad ,$$

where

$$R^{r_l} = \sum_{m=1}^{N} R^{f_m} f_m^{r_l} \quad . \tag{E.5.5}$$

Our next steps consist in the characterization of the Jacobian matrix (E.5.1) in the fixed point, computation of the characteristic polynomial and, finally, analysis of its roots. Since there are two general types of the equilibria, we separate our further analysis. First, we consider the case with $r^* \neq -\bar{e}$ and prove Propositions 5.3.3 and 5.3.4. Then, we move to the case where $r^* = -\bar{e}$ and eventually prove Proposition 5.3.5.

E.5.1 Equilibria with $r^* \neq -\bar{e}$.

Let $x_{1\diamond k}$ denote the equilibrium investment shares of survivor(s) in such equilibrium. The Jacobian structure in the fixed point is determined by the values of derivatives of functions providing wealth shares and return. We compute them in the following

Lemma E.5.1. *Consider equilibrium $\boldsymbol{x^*}$ of system* (5.3.3) *with $k \geq 1$ survivors and $r^* \neq -\bar{e}$. In this equilibrium functions Φ_n defined in* (5.3.5) *for all $n \in \{1, \dots, N-1\}$ have the following derivatives:*

$$\begin{aligned}
\Phi_n^{x_m} &= \varphi_n^* \left(\delta_{n,m} - \varphi_m^*\right) \frac{\bar{e} + r^*}{1 + r^*} && \forall m \in \{1, \dots, N\} \quad , \\
\Phi_n^{\varphi_m} &= \frac{\delta_{n,m}\big(1 + x_n^*(r^* + \bar{e})\big) - \varphi_n^*(r^* + \bar{e})(x_m^* - x_N^*)}{1 + r^*} && \forall m \in \{1, \dots, N-1\} \quad , \\
\Phi_n^{R} &= \varphi_n^* \frac{x_n^* - x_{1\diamond k}^*}{1 + r^*} \quad ,
\end{aligned} \tag{E.5.6}$$

where $\delta_{n,m}$ is the Kronecker delta. Furthermore, in this equilibrium function R defined in (5.3.4) *has the following derivatives:*

$$\begin{aligned}
R^{x_m} &= -\, \varphi_m^* \frac{1}{x_{1\diamond k}^* \left(1 - x_{1\diamond k}^*\right)} && \forall m \in \{1, \dots, N\} \quad , \\
R^{f_m} &= \varphi_m^* \frac{1 + r^*}{x_{1\diamond k}^* \left(1 - x_{1\diamond k}^*\right)} && \forall m \in \{1, \dots, N\} \quad , \\
R^{\varphi_m} &= \frac{r^* + \bar{e}}{x_{1\diamond k}^* \left(1 - x_{1\diamond k}^*\right)} \big(x_m^* \left(x_m^* - x_{1\diamond k}^*\right) - x_N^* \left(x_N^* - x_{1\diamond k}^*\right)\big) && \forall m \in \{1, \dots, N-1\} \, .
\end{aligned} \tag{E.5.7}$$

This statement is essentially identical to Lemma D.7.1 and can be checked by the direct computations.

With results of Lemma E.5.1 one can simplify the Jacobian (E.5.1). The following applies:

Lemma E.5.2. *Let $\boldsymbol{x^*}$ be an EML equilibrium of system* (5.3.3). *The Jacobian matrix computed in this point $\boldsymbol{J}(\boldsymbol{x^*})$ has the following structure*

$$\left\| \begin{array}{cccccc|cccccc|cccc}
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & f_N^{r_0} & \dots & f_N^{r_{L-2}} & f_N^{r_{L-1}} \\
\hline
\Phi_1^{x_1} & \dots & \Phi_1^{x_k} & 0 & \dots & 0 & \Phi_1^{\varphi_1} & \dots & \Phi_1^{\varphi_k} & \Phi_1^{\varphi_{k+1}} & \dots & \Phi_1^{\varphi_{N-1}} & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\Phi_k^{x_1} & \dots & \Phi_k^{x_k} & 0 & \dots & 0 & \Phi_k^{\varphi_1} & \dots & \Phi_k^{\varphi_k} & \Phi_k^{\varphi_{k+1}} & \dots & \Phi_k^{\varphi_{N-1}} & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \Phi_{k+1}^{\varphi_{k+1}} & \dots & 0 & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \Phi_{N-1}^{\varphi_{N-1}} & 0 & \dots & 0 & 0 \\
\hline
R^{x_1} & \dots & R^{x_k} & 0 & \dots & 0 & R^{\varphi_1} & \dots & R^{\varphi_k} & R^{\varphi_{k+1}} & \dots & R^{\varphi_{N-1}} & R^{r_0} & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0
\end{array} \right\|$$

Proof. From (E.5.6) and (5.3.9) and (5.3.10) it follows that $\Phi_n^R = 0$ for any agent n. Therefore, block $[\partial\mathcal{W}/\partial\mathcal{R}]$ is zero matrix, and also

$$\left[\frac{\partial \mathcal{W}}{\partial \mathcal{X}}\right]_{n,m} = \begin{cases} \Phi_n^{x_m} & m, n \leq k \\ 0 & \text{otherwise} \end{cases} \quad , \qquad \left[\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\right]_{n,m} = \begin{cases} 0 & n > k \ , \quad n \neq m \\ \Phi_n^{\varphi_m} & \text{otherwise} \end{cases}$$

From (E.5.7) it follows that for that $R^{x_m} = R^{f_m} = 0$ for $m > k$, i.e. for those who do not survive. The structure above immediately follows. Notice, moreover, that in definition of R^{r_l} in (E.5.5) only k first terms are present. $\square$

Lemma E.5.3. *The characteristic polynomial P_J of the matrix $\boldsymbol{J}(\boldsymbol{x}^*)$ can be reduced to the following form*

$$P_J(\mu) = (-1)^{N+L}\,\mu^{N-1}\,(1-\mu)^{k-1}\prod_{j=k+1}^{N}\left(\frac{1+x_j^*\,(r^*+\bar{e})}{1+r^*}-\mu\right)\left(\mu^{L+1}-\frac{(1+r^*)\mu-1}{x^*_{1\diamond k}(1-x^*_{1\diamond k})}\sum_{j=1}^{k}\varphi_j^*\,P_{f_j}(\mu)\right) \tag{E.5.8}$$

where P_{f_n} is the stability polynomial associated to the n-th investment function as defined in (5.2.5).

Proof. The following proof is constructive: we will identify in succession the factors appearing in (E.5.8). At each step, a set of eigenvalues is found and the problem is reduced to the analysis of the residual matrix obtained removing the rows and columns associated with the relative eigenspace. In this way the dimension of the analyzed matrix is progressively reduced.

Consider the Jacobian matrix in Lemma E.5.2. The last $N-k$ columns of the left blocks contain only zero entries so that the matrix possesses eigenvalue 0 with (at least) multiplicity $N-k$. Moreover, in each of the last $N-1-k$ rows in the central blocks the only non-zero entries are on the main diagonal. Consequently, $\Phi_j^{\varphi_j}$ for $k+1 \le j \le N-1$ are eigenvalues of the matrix, with multiplicity (at least) one. A first contribution to the characteristic polynomial is then determined as

$$(-\mu)^{N-k}\prod_{j=k+1}^{N-1}(\Phi_j^{\varphi_j}-\mu) = (-\mu)^{N-k}\prod_{j=k+1}^{N-1}\left(\frac{1+x_j^*(r^*+\bar{e})}{1+r^*}-\mu\right) \tag{E.5.9}$$

where we used (E.5.6) to compute $\Phi_j^{\varphi_j}$ at equilibrium.

In order to find the remaining part of the characteristic polynomial we eliminate the rows and columns associated to the previous eigenvalues to obtain

$$\left\|\begin{array}{ccc|ccc|cccc}
0 & \dots & 0 & 0 & \dots & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & f_k^{r_0} & \dots & f_k^{r_{L-2}} & f_k^{r_{L-1}} \\
\hline
\Phi_1^{x_1} & \dots & \Phi_1^{x_k} & \Phi_1^{\varphi_1} & \dots & \Phi_1^{\varphi_k} & 0 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\Phi_k^{x_1} & \dots & \Phi_k^{x_k} & \Phi_k^{\varphi_1} & \dots & \Phi_k^{\varphi_k} & 0 & \dots & 0 & 0 \\
\hline
R^{x_1} & \dots & R^{x_k} & R^{\varphi_1} & \dots & R^{\varphi_k} & R^{r_0} & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\
0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0
\end{array}\right\| \tag{E.5.10}$$

The remaining term in the characteristic polynomial of the original system is represented by the characteristic polynomial of the latter matrix, which we call $\boldsymbol{L}$. This quadratic matrix has $2k+L$ rows when $k < N$. If $k = N$, representation (E.5.10) is, strictly speaking, not correct. Indeed, there exist only $N-1$ wealth shares φ's in the original system, therefore the central block of the matrix has maximal dimension $(N-1)\times(N-1)$. Therefore, in this case, the correct matrix has dimension $(2N+L-1)\times(2N+L-1)$ and can be obtained from (E.5.10) through the elimination of the last row and the last column in the central blocks. We will compute now the characteristic polynomial, i.e. determinant $\det(\boldsymbol{L}-\lambda\boldsymbol{I})$, where $\boldsymbol{I}$ denotes an identity matrix of the corresponding dimension. We consider separately the following two cases: when $k<N$ and when $k=N$.

If $k<N$, then from (E.5.6) it follows that for $n,m\le k$ it is

$$\Phi_n^{\varphi_m} = \begin{cases} 1-\varphi_n^* v & \text{if } n=m \\ -\varphi_n^* v & \text{otherwise} \end{cases}, \qquad \text{where} \qquad v = \left(x^*_{1\diamond k} - x^*_N\right)\frac{\bar{e}+r^*}{1+r^*}\,. \tag{E.5.11}$$

Moreover, since all survivors invest share $x_{1\diamond k}$, it follows from (E.5.7) that for $m\le k$

$$R^{\varphi_m} = v\,b\,, \qquad \text{where} \qquad b = x^*_N\,\frac{1+r^*}{x^*_{1\diamond k}\,(1-x^*_{1\diamond k})}\quad. \tag{E.5.12}$$

Thus, almost all elements in the central column block of (E.5.10) are proportional to the constant v. It suggests that for the computation of the characteristic polynomial of this matrix, one can represent each of these column as appropriate sum and then use the multilinear property of the determinant. In order to implement this idea, we introduce the following column vectors

$$\begin{aligned}
\boldsymbol{b} &= \left\| \begin{array}{ccc|ccc|cccc} 0 & \dots & 0 & -\varphi_1^* & \dots & -\varphi_k^* & b & 0 & \dots & 0 \end{array} \right\|\,,\\
\boldsymbol{b}_1 &= \left\| \begin{array}{ccc|ccc|cccc} 0 & \dots & 0 & 1-\mu & \dots & 0 & 0 & 0 & \dots & 0 \end{array} \right\|\,,\\
&\ \ \vdots\\
\boldsymbol{b}_k &= \left\| \begin{array}{ccc|ccc|cccc} 0 & \dots & 0 & 0 & \dots & 1-\mu & 0 & 0 & \dots & 0 \end{array} \right\|\,.
\end{aligned}$$

The central column block in the determinant $\det(\boldsymbol{L}-\lambda\boldsymbol{I})$ can now be represented as

$$\left\| \, v\,\boldsymbol{b}+\boldsymbol{b}_1 \mid \dots \mid v\,\boldsymbol{b}+\boldsymbol{b}_k \, \right\| \quad,$$

and we apply the multilinear property to this block. Thus, we consider each of these columns as a sum of two terms and end up with a sum of 2^k determinants. Notice, however, that many of them are zeros, since they contain two or more columns proportional to vector $\boldsymbol{b}$. There are only $k+1$ non-zero elements in the expansion. One of them has the following structure of the central column block:

$$\left\| \, \boldsymbol{b}_1 \mid \dots \quad \dots \quad \dots \mid \boldsymbol{b}_k \, \right\| \quad,$$

while k others possess similar structure in the central column block, with column $v\,\boldsymbol{b}$ on the ν'th place instead of $\boldsymbol{b}_\nu$ for all $\nu\in\{1,\dots,k\}$:

$$\left\| \, \boldsymbol{b}_1 \mid \dots \mid v\,\boldsymbol{b} \mid \dots \mid \boldsymbol{b}_k \, \right\| \quad.$$

Matrix with the former block contains diagonal block in the center corner and, therefore, its determinant is equal to $(1-\mu)^k \det \boldsymbol{M}(k)$, where

$$\boldsymbol{M}(k) = \left\| \begin{array}{ccc|cccc}
-\mu & \dots & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & -\mu & f_k^{r_0} & \dots & f_k^{r_{L-2}} & f_k^{r_{L-1}} \\
\hline
R^{x_1} & \dots & R^{x_k} & R^{r_0}-\mu & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\
0 & \dots & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 1 & -\mu
\end{array} \right\| \tag{E.5.13}$$

Other k determinants can be simplified in analogous way, so that

$$\det(\boldsymbol{L}-\lambda\boldsymbol{I}) = (1-\mu)^k \det \boldsymbol{M}(k) + (1-\mu)^{k-1} \sum_{\nu=1}^{k} \det \boldsymbol{M}_\nu(k) \quad , \tag{E.5.14}$$

where for all $\nu \in \{1,\dots,k\}$ we define the following matrix

$$\boldsymbol{M}_\nu(k) = \left\| \begin{array}{ccc|c|cccc} -\mu & \dots & 0 & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & -\mu & 0 & f_k^{r_0} & \dots & f_k^{r_{L-2}} & f_k^{r_{L-1}} \\ \hline \Phi_\nu^{x_1} & \dots & \Phi_\nu^{x_k} & -v\,\varphi_\nu^* & 0 & \dots & 0 & 0 \\ \hline R^{x_1} & \dots & R^{x_k} & v\,b & R^{r_0}-\mu & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \end{array} \right\| \ . \tag{E.5.15}$$

To simplify the determinant of the last matrix, we expand it on the minors of the elements of the central column. For this purpose we for each $\nu \in \{1,\dots,k\}$ introduce yet another matrix whose structure is similar with $\boldsymbol{M}(k)$:

$$\boldsymbol{N}_\nu(k) = \left\| \begin{array}{ccc|ccccc} -\mu & \dots & 0 & f_1^{r_0} & f_1^{r_1} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & -\mu & f_k^{r_0} & f_k^{r_1} & \dots & f_k^{r_{L-2}} & f_k^{r_{L-1}} \\ \hline \Phi_\nu^{x_1} & \dots & \Phi_\nu^{x_k} & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 & -\mu & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu \end{array} \right\| \ .$$

Then it is:

$$\det \boldsymbol{M}_\nu(k) = v\big(-\varphi_\nu^* \det \boldsymbol{M}(k) - b \det \boldsymbol{N}_\nu(k)\big) \quad . \tag{E.5.16}$$

We compute the determinant of matrix $\boldsymbol{M}(k)$ in a recursive way. Consider the expansion by the determinants of the minors of the elements of the first column. The minor of the first element $-\mu$ is a matrix with a structure similar to $\boldsymbol{M}(k)$. This is the final matrix obtained if the first agents were not among the survivors. Let us denote its determinant with $\boldsymbol{M}(k-1)$. The minor associated with R^{x_1} has a left upper block with $k-1$ entries equal to $-\mu$ below the main diagonal. This block generates a contribution μ^{k-1} to the determinant and once its columns and rows are eliminated, one remains with a matrix of the type in (E.1.1). Applying Lemma E.1.1 one then has

$$\det \boldsymbol{M}(k) = (-\mu)\,\det \boldsymbol{M}(k-1) + (-1)^k R^{x_1} \mu^{k-1}\,(-1)^{L-1}\,P_{f_1}(\mu)\ ,$$

where P_{f_1} is the stability polynomial associated with the first investment function. Applying recursively the relation above, the dimension of the determinant is progressively reduced. At the end one remains with the lower right block of the original matrix, which is again a matrix similar to (E.1.1). Applying once more Lemma E.1.1 one has for $\boldsymbol{M}(k)$ the following

$$\det \boldsymbol{M}(k) = (-1)^{L-1+k}\,\mu^{k-1} \sum_{j=1}^{k} R^{x_j}\,P_{f_j}(\mu) + (-1)^{L-1+k}\,\mu^k \left(\sum_{j=0}^{L-1} R^{r_j} \mu^{L-1-j} - \mu^L \right) \ . \tag{E.5.17}$$

The determinant of matrix $\boldsymbol{N}_\nu(k)$ can be computed using the similar strategy. The only difference is that in the last recursive step one of the matrix has zero determinant. Therefore, we have:

$$\det \boldsymbol{N}_\nu(k) = (-\mu)\,\det \boldsymbol{N}_\nu(k-1) + (-1)^k \Phi_\nu^{x_1} \mu^{k-1}\,(-1)^{L-1}\,P_{f_1}(\mu) =$$
$$= (-1)^{L-1+k}\,\mu^{k-1}\sum_{j=1}^{k} \Phi_\nu^{x_j}\,P_{f_j}(\mu) \quad ,$$

which, taking into account (E.5.16), implies

$$\sum_{\nu=1}^{k} \det \boldsymbol{M}_\nu(k) = v\left(-\det \boldsymbol{M}(k) - b\sum_{\nu=1}^{k}\det \boldsymbol{N}_\nu(k)\right) =$$
$$= -v\,\det \boldsymbol{M}(k) + v\,b\,(-1)^{L+k}\,\mu^{k-1}\sum_{\nu=1}^{k}\sum_{j=1}^{k}\Phi_\nu^{x_j}\,P_{f_j}(\mu) = -v\,\det \boldsymbol{M}(k) \quad .$$

The last equality above follows directly from expression for $\Phi_\nu^{x_j}$ in (E.5.6).

Substitution of the last relation into (E.5.14), use of the expression for $\det \boldsymbol{M}(k)$ from (E.5.17) where the corresponding values of the derivatives of function R are computed in accordance with (E.5.7), leads to the last contribution into characteristic polynomial:

$$\det(\boldsymbol{L} - \lambda \boldsymbol{I}) = (1-\mu)^{k-1}\,(1-\mu-v)\,\det \boldsymbol{M}(k) =$$
$$= (-1)^{L-1+k}\,\mu^{k-1}\,(1-\mu)^{k-1}\left(\frac{1+x_N^*(r^*+\bar{e})}{1+r^*} - \mu\right)$$
$$\left(\frac{(1+r^*)\mu - 1}{x_{1\diamond k}^*(1-x_{1\diamond k}^*)}\sum_{j=1}^{k}\varphi_j^*\,P_{f_j}(\mu) - \mu^{L+1}\right) \quad . \tag{E.5.18}$$

If $k = N$, i.e. all agents survive, then all investment shares are the same. In this case, according to Lemma E.5.1, all elements in the central column block of matrix (E.5.10) are zeros apart from the ones on the diagonal in the central $(N-1)\times(N-1)$ matrix. It contributes to the characteristic polynomial by the factor $(1-\mu)^{N-1}$. The remaining part is the determinant of matrix $\boldsymbol{M}$ in this case. This is consistent with (E.5.18).

The product of (E.5.9) and (E.5.18) gives (E.5.8), what completes the proof. □

Using the characteristic polynomial of the Jacobian matrix it is straightforward to derive the equilibrium stability conditions for equilibria with $r^* \neq -\bar{e}$.

Case of one survivor: Proof of Proposition 5.3.3

If $k = 1$ the characteristic polynomial (E.5.8) reduces to

$$P_J(\mu) = (-1)^{N+L}\,\mu^{N-1}\prod_{j=2}^{N}\left(\frac{1+x_j^*\,(r^*+\bar{e})}{1+r^*} - \mu\right)\left(\mu^{L+1} - \frac{(1+r^*)\mu - 1}{x_1^*(1-x_1^*)}\,P_{f_1}(\mu)\right) \; .$$

From the expression of the derivative of the EML at equilibrium $l'(r^*)$ one can see that last factor corresponds to the polynomial Q_1 in (5.3.16). The conditions in (5.3.17) are derived from the requirement

$$\left|\frac{1+x_j^*\,(r^*+\bar{e})}{1+r^*}\right| < 1 \qquad j > 1\,,$$

and the Proposition is proved.

Case of many survivors: Proof of Proposition 5.3.4

In the case of $k > 1$ survivors the characteristic polynomial in (E.5.8) possesses a unit root with multiplicity $k-1$. Consequently, the fixed point is non-hyperbolic.

Let us find the eigenspace associated to eigenvalue 1. We subtract from Jacobian matrix (E.5.1) computed at equilibrium the identity matrix of the corresponding dimension and analyze the kernel of the resulting $\boldsymbol{J}-\boldsymbol{I}$ matrix. In order to do it we proceed to the reduced Jacobian (E.5.10) and apply the same procedure. We consider the $k < N$ and the $k = N$ cases separately.

When $k < N$, as we showed in the proof of Lemma E.5.3, in the matrix obtained as a result of subtraction of an identity matrix from (E.5.10), the central $k-1$ columns are identical, see (E.5.11) and (E.5.12). Therefore, the kernel of the matrix $\boldsymbol{J}-\boldsymbol{I}$ can be generated by a basis containing the following $k-1$ vectors

$$\boldsymbol{u}_n = \Big(\underbrace{0,\dots,0}_{N};\underbrace{0,\dots,0}_{n-1},1,\underbrace{0,\dots,0}_{k-n-1},-1;\underbrace{0,\dots,0}_{N-1-k};\underbrace{0,\dots,0}_{L}\Big)\,, \quad 1 \le n \le k-1 \quad . \tag{E.5.19}$$

Notice that the direction of vector $\boldsymbol{u}_n$ corresponds to a change in the relative wealths of the n-th and k-th survivor.

If, instead, $k = N$, then the last $k-1$ columns in the resulting (from (E.5.10)) matrix are zero vectors, and then the kernel of the matrix $\boldsymbol{J}-\boldsymbol{I}$ can be generated with the $N-1$ vectors of the canonical basis

$$\boldsymbol{v}_n = \Big(\underbrace{0,\dots,0}_{N};\underbrace{0,\dots,0}_{n-1},1,\underbrace{0,\dots,0}_{N-n-1};\underbrace{0,\dots,0}_{L}\Big)\,, \quad 1 \le n \le N-1 \quad . \tag{E.5.20}$$

whose direction corresponds to a change in the relative wealths of the n-th and N-th survivors.

If the system is perturbed away from equilibrium $\boldsymbol{x}^*$ along the directions defined in (E.5.19) or (E.5.20), a new fixed point is reached. Then, the system is stable, but not asymptotically stable, with respect to these perturbations.

Moreover, since the eigenspaces identified above do not depend on the system parameters, it is immediate to realize that they do constitute not only the tangent spaces to the corresponding non-hyperbolic manifolds, but the manifolds themselves.

The polynomial (5.3.18) is the last factor in (E.5.8), while conditions (5.3.19) are obtained by imposing

$$\left|\frac{1+x_j^*\,(r^*+\bar{e})}{1+r^*}\right| < 1 \qquad j > k+1\,,$$

which completes the proof.

E.5.2 Equilibria with $r^* = -\bar{e}$.

Let us now move to those equilibria which we found in Proposition 5.3.2 and suppose that $k \le N$ agents survive in this equilibrium. As usually, the Jacobian structure in the fixed point can be established by the evaluation of derivatives of corresponding functions. We provide these derivatives in the following

Lemma E.5.4. *Consider equilibrium $\boldsymbol{x}^*$ of system (5.3.3) with k survivors and $r^* = -\bar{e}$. In this equilibrium functions Φ_n defined in (5.3.5) for all $n \in \{1,\dots,N-1\}$ have the following derivatives:*

$$\begin{aligned}
&\Phi_n^{x_m} = 0 && \forall m \in \{1,\dots,N\} \quad ,\\
&\Phi_n^{\varphi_m} = \delta_{n,m} && \forall m \in \{1,\dots,N-1\} \quad ,\\
&\Phi_n^{R} \;= \varphi_n^*\, x_n^* \quad ,
\end{aligned} \tag{E.5.21}$$

where $\delta_{n,m}$ is the Kronecker delta.

Furthermore, in this equilibrium function R defined in (5.3.4) *has the following derivatives:*

$$
\begin{aligned}
R^{x_m} &= \varphi_m^* \frac{1-\bar{e}}{\langle x^2 \rangle} \qquad \forall m \in \{1,\dots,N\} \quad , \\
R^{f_m} &= -\varphi_m^* \frac{1}{\langle x^2 \rangle} \qquad \forall m \in \{1,\dots,N\} \quad , \\
R^{\varphi_m} &= -\bar{e}\, \frac{x_m^* - x_N^*}{\langle x^2 \rangle} \qquad \forall m \in \{1,\dots,N-1\} \quad .
\end{aligned}
\tag{E.5.22}
$$

where $\langle x^2 \rangle$ denotes the weighted sum of squares of the equilibrium investment shares:

$$
\langle x^2 \rangle = \sum_{j=1}^{k} \varphi_j^* {x_j^*}^2 \quad . \tag{E.5.23}
$$

This Lemma repeats Lemma D.7.4 and can be checked by the direct differentiation.

With results of Lemma E.5.4 one can simplify the Jacobian (E.5.1). The following applies:

Lemma E.5.5. *Let $\boldsymbol{x}^*$ be a no-arbitrage equilibrium of system* (5.3.3)*. The Jacobian matrix computed in this point $\boldsymbol{J}(\boldsymbol{x}^*)$ has the following structure*

$$
\left\| \begin{array}{cccccc|cccccc|ccc}
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & f_1^{r_0} & \cdots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & f_N^{r_0} & \cdots & f_N^{r_{L-2}} & f_N^{r_{L-1}} \\
\hline
\Phi_1^R R^{x_1} & \cdots & \Phi_1^R R^{x_k} & 0 & \cdots & 0 & 1+\Phi_1^R R^{\varphi_1} & \cdots & \Phi_1^R R^{\varphi_k} & \Phi_1^R R^{\varphi_{k+1}} & \cdots & \Phi_1^R R^{\varphi_{N-1}} & \Phi_1^{r_0} & \cdots & \Phi_1^{r_{L-2}} & \Phi_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\Phi_k^R R^{x_1} & \cdots & \Phi_k^R R^{x_k} & 0 & \cdots & 0 & \Phi_k^R R^{\varphi_1} & \cdots & 1+\Phi_k^R R^{\varphi_k} & \Phi_k^R R^{\varphi_{k+1}} & \cdots & \Phi_k^R R^{\varphi_{N-1}} & \Phi_k^{r_0} & \cdots & \Phi_k^{r_{L-2}} & \Phi_k^{r_{L-1}} \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\
\hline
R^{x_1} & \cdots & R^{x_k} & 0 & \cdots & 0 & R^{\varphi_1} & \cdots & R^{\varphi_k} & R^{\varphi_{k+1}} & \cdots & R^{\varphi_{N-1}} & R^{r_0} & \cdots & R^{r_{L-2}} & R^{r_{L-1}} \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0
\end{array} \right\|
$$

where $\Phi_n^{r_l} = \Phi_n^R \cdot \sum_{m=1}^k R^{f_m} f_m^{r_l}$ in accordance with (E.5.4).

Proof. The results of Lemma E.5.5 imply that for all agents (with corresponding indeces) who do not survive in the equilibrium $\Phi_n^R = 0$, $R^{x_m} = 0$ and $R^{f_m} = 0$. Therefore, the last row of the blocks of the Jacobian can be simplified and also

$$
\left[\frac{\partial \mathcal{W}}{\partial \mathcal{X}}\right]_{n,m} = \begin{cases} \Phi_n^R R^{x_m} & m,n \le k \\ 0 & \text{otherwise} \end{cases} \quad , \qquad \left[\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\right]_{n,m} = \begin{cases} \delta_{n,m} + \Phi_n^R R^{\varphi_m} & n \le k \\ \delta_{n,m} & \text{otherwise} \end{cases}
$$

Moreover, the last row of the blocks of the Jacobian can be also simplified. □

Lemma E.5.6. *Characteristic polynomial P_J of matrix $\boldsymbol{J}(\boldsymbol{x}^*)$ in the equilibrium with $r^* = -\bar{e}$ reads*

$$
P_J(\mu) = (-1)^{L+N} (1-\mu)^{N-2} \mu^{N-1} \left(\mu + \bar{e} - 1\right) \left(\mu^{L+1} + \frac{\mu - 1}{\langle x^2 \rangle} \sum_{j=1}^{k} \varphi_j^* P_{f_j}(\mu) \right) \tag{E.5.24}
$$

Proof. The proof is constructive and analogous to the proof of Lemma E.5.3. From the Jacobian matrix in Lemma E.5.5 we can immediately identify that in each of the $N-1-k$ last rows belonging to the central row clock of the matrix the only non-zero entries belong to the main diagonal of $[\partial \mathcal{W}/\partial \mathcal{W}]$ and equal to 1. In addition, the last $N-k$ columns of the leftmost blocks contain only zero entries. Together, it gives the first entry in the characteristic polynomial:

$$(-\mu)^{N-k}\,(1-\mu)^{N-1-k} \quad . \tag{E.5.25}$$

(If all N agents survive in equilibrium the second term in the expression above is absent.) In order to find the remaining factors we eliminate the rows and columns associated to the previous eigenvalues. The matrix which we obtain after such elimination reads:

$$\left\| \begin{array}{ccc|ccc|cccc}
0 & \dots & 0 & 0 & \dots & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & f_N^{r_0} & \dots & f_N^{r_{L-2}} & f_N^{r_{L-1}} \\
\hline
\varPhi_1^R R^{x_1} & \dots & \varPhi_1^R R^{x_k} & 1+\varPhi_1^R R^{\varphi_1} & \dots & \varPhi_1^R R^{\varphi_k} & \varPhi_1^{r_0} & \dots & \varPhi_1^{r_{L-2}} & \varPhi_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\varPhi_k^R R^{x_1} & \dots & \varPhi_k^R R^{x_k} & \varPhi_k^R R^{\varphi_1} & \dots & 1+\varPhi_k^R R^{\varphi_k} & \varPhi_k^{r_0} & \dots & \varPhi_k^{r_{L-2}} & \varPhi_k^{r_{L-1}} \\
\hline
R^{x_1} & \dots & R^{x_k} & R^{\varphi_1} & \dots & R^{\varphi_k} & R^{r_0} & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\
0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0
\end{array} \right\| \tag{E.5.26}$$

Notice that this matrix, as it is written, has dimension $(2k+L)\times(2k+L)$. Strictly speaking, this is not always correct. The problematic case is one with $k=N$, when the correct matrix has dimension $(2N-1+L)\times(2N-1+L)$ due to the fact that there are only $N-1$ wealth shares. Thus, when $k=N$ the block in the central row and central column has a size $(N-1)\times(N-1)$. In our further computations there are no big differences between cases $k<N$ and $k=N$. These computations are, formally, valid for the case $k<N$, however. Corresponding differences for $k=N$ we mention in the footnotes. We denote matrix (E.5.26), shortened if $k=N$, as $\boldsymbol{L}$ and compute the remaining term in the characteristic polynomial of the original system as $\det(\boldsymbol{L}-\lambda\boldsymbol{I})$, where $\boldsymbol{I}$ is an identity matrix of the corresponding dimension.

We apply the multilinear property of the determinant to the central block of columns in matrix (E.5.26). In order to implement this idea, we introduce the following column vectors of the length[1] $2k+L$:

$$\begin{aligned}
\boldsymbol{d} &= \left\| \begin{array}{ccc|ccc|cccc} 0 & \dots & 0 & \varPhi_1^R & \dots & \varPhi_k^R & 1 & 0 & \dots & 0 \end{array} \right\| \quad , \\
\boldsymbol{d}_1 &= \left\| \begin{array}{ccc|ccc|cccc} 0 & \dots & 0 & 1-\mu & \dots & 0 & 0 & 0 & \dots & 0 \end{array} \right\| \quad , \\
&\ \, \vdots \\
\boldsymbol{d}_k &= \left\| \begin{array}{ccc|ccc|cccc} 0 & \dots & 0 & 0 & \dots & 1-\mu & 0 & 0 & \dots & 0 \end{array} \right\| \quad .
\end{aligned}$$

The central column block in the determinant $\det(\boldsymbol{L}-\lambda\boldsymbol{I})$ can now be represented as

$$\left\| \, R^{\varphi_1}\,\boldsymbol{d}+\boldsymbol{d}_1 \mid \dots \mid R^{\varphi_k}\,\boldsymbol{d}+\boldsymbol{d}_k \, \right\| \quad ,$$

and we apply the multilinear property to this block. Thus, we consider each of these columns as a sum of two terms and end up with a sum of 2^k determinants. Notice, however, that many of them

[1]Of course, when $k=N$ agents survive, these vectors have length $2N-1+L$ each. Moreover, in this case there is vector $\boldsymbol{d}$ and only $N-1$ vectors $\boldsymbol{d}_1 \dots \boldsymbol{d}_{N-1}$.

are zeros, since they contain two or more columns proportional to vector $\boldsymbol{d}$. There are only $k+1$ non-zero elements in the expansion[2]. One of them has the following structure of the last column block:

$$\left\| \, \boldsymbol{d}_1 \mid \dots \quad \dots \quad \dots \mid \boldsymbol{d}_k \, \right\| \quad ,$$

while k others possess similar structure in the last column block, with column $R^{\varphi_\nu}\,\boldsymbol{d}$ on the ν'th place instead of $\boldsymbol{d}_\nu$ for all $\nu \in \{1, \dots, k\}$:

$$\left\| \, \boldsymbol{d}_1 \mid \dots \mid R^{\varphi_\nu}\,\boldsymbol{d} \mid \dots \mid \boldsymbol{d}_k \, \right\| \quad .$$

Matrix with the former block contains the diagonal central and, therefore, its determinant is equal to $(1-\mu)^k \det \boldsymbol{N}(k)$, where matrix $\boldsymbol{N}(k)$ is identical to the matrix $\boldsymbol{M}(k)$ defined in (E.5.13). We use here another notation, however, in order to stress that the partial derivatives R^{x_j} and R^{f_j} used in these two matrices have different values in different equilibria. For the equilibrium which we consider here, these derivatives were computed in Lemma E.5.4. Using (E.5.17) it is, therefore, immediate to see that

$$\det \boldsymbol{N}(k) = (-1)^{L+k}\,\mu^{k-1}\left(\mu^{L+1} + \frac{\mu+\bar{e}-1}{\langle x^2\rangle}\sum_{j=1}^{k}\varphi_j^*\,P_{f_j}(\mu)\right) \quad . \tag{E.5.27}$$

Analogously, other k determinants can be represented as $(1-\mu)^{k-1}\det \boldsymbol{N}_\nu(k)$, where for all $\nu \in \{1, \dots, k\}$ we define matrix

$$\boldsymbol{N}_\nu(k) = \left\| \begin{array}{ccc|c|cccc}
-\mu & \dots & 0 & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & -\mu & 0 & f_k^{r_0} & \dots & f_k^{r_{L-2}} & f_k^{r_{L-1}} \\
\hline
\Phi_\nu^R R^{x_1} & \dots & \Phi_\nu^R R^{x_k} & R^{\varphi_\nu}\Phi_v^R & \Phi_\nu^{r_0} & \dots & \Phi_\nu^{r_{L-2}} & \Phi_\nu^{r_{L-1}} \\
\hline
R^{x_1} & \dots & R^{x_k} & R^{\varphi_\nu} & R^{r_0}-\mu & \dots & R^{r_{L-2}} & R^{r_{L-1}} \\
0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu
\end{array} \right\| \; .$$

This matrix can be immediately simplified, since its central row is (almost) proportional to the next row (the first row in the bottom block). Applying multilinear property of the determinant, we get

$$\boldsymbol{N}_\nu(k) = \left\| \begin{array}{ccc|c|cccc}
-\mu & \dots & 0 & 0 & f_1^{r_0} & \dots & f_1^{r_{L-2}} & f_1^{r_{L-1}} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & -\mu & 0 & f_k^{r_0} & \dots & f_k^{r_{L-2}} & f_k^{r_{L-1}} \\
\hline
\Phi_\nu^R R^{x_1} & \dots & \Phi_\nu^R R^{x_k} & R^{\varphi_\nu}\Phi_v^R & \Phi_\nu^{r_0} & \dots & \Phi_\nu^{r_{L-2}} & \Phi_\nu^{r_{L-1}} \\
\hline
0 & \dots & 0 & 0 & -\mu & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & 1 & -\mu
\end{array} \right\| \; .$$

The determinant of this matrix can be easily computed. It reads

$$\det \boldsymbol{N}_\nu(k) = (-\mu)^{L+k} R^{\varphi_\nu}\Phi_v^R \quad .$$

[2] Correspondingly, if $k = N$ there are only N elements in the expansion. In particular, index ν introduced below belongs to the set $\{1, \dots, N-1\}$ in this case.

Using the expressions for $\varPhi_\nu^R$ and R^{φ_ν} from (E.5.21) and (E.5.22), one can directly check[3] that $\sum_{\nu=1}^k \varPhi_\nu^R R^{\varphi_\nu} = -\bar{e}$. Therefore,

$$\det(\boldsymbol{L} - \lambda \boldsymbol{I}) = (1-\mu)^k \det \boldsymbol{N}(k) + (1-\mu)^{k-1} \sum_{\nu=1}^k \det \boldsymbol{N}_\nu(k) =$$

$$= (1-\mu)^k \det \boldsymbol{N}(k) - (1-\mu)^{k-1} (-\mu)^{L+k} \bar{e} =$$

$$= (1-\mu)^{k-1} (-1)^{L+k} \mu^{k-1}$$

$$\left((1-\mu)\mu^{L+1} + (1-\mu) \frac{\mu + \bar{e} - 1}{\langle x^2 \rangle} \sum_{j=1}^k \varphi_j^* P_{f_j}(\mu) - \mu^{L+1} \bar{e} \right) =$$

$$= (1-\mu)^{k-1} (-1)^{L+k} \mu^{k-1} (\mu + \bar{e} - 1) \left(\mu^{L+1} + \frac{\mu - 1}{\langle x^2 \rangle} \sum_{j=1}^k \varphi_j^* P_{f_j}(\mu) \right) \tag{E.5.28}$$

Combining now (E.5.25) and (E.5.28) we get the polynomial (E.5.24). □

Using the characteristic polynomial of the Jacobian matrix it is straightforward to derive the equilibrium stability conditions for equilibria with $r^* = -\bar{e}$.

Proof of Proposition 5.3.5

Independently of the number of survivors, the characteristic polynomial in (E.5.24) possesses a unit root with multiplicity $N-2$. Consequently, the fixed point $\boldsymbol{x}^*$ is never hyperbolic, when $N \geq 3$.

In Corollary 5.3.2 we have found that there is a manifold of equilibria of dimension $N-2$. Moving from $\boldsymbol{x}^*$ along this manifold we reach new fixed points. Therefore, this manifold represents non-hyperbolic manifold of the fixed point $\boldsymbol{x}^*$. For the stability of equilibrium $\boldsymbol{x}^*$ with respect to the perturbations in the directions *orthogonal* to this manifold, it is sufficient to have all other eigenvalues inside the unit circle. If this condition is satisfied, then equilibrium $\boldsymbol{x}^*$ of the system is stable, but not asymptotically stable.

Since $\bar{e} > 0$, this sufficient condition can be expressed through the roots of the last term in (E.5.24). This term is exactly polynomial (5.3.21).

[3] This is the place where the difference between the case with $k < N$ and $k = N$ matters. The point is that in the latter situation there are only $N-1$ matrices $\boldsymbol{N}_\nu$. However, the result of the summation is the same in both cases.

Appendix F

Proofs of Propositions in Chapter 6

F.1 Proof of Proposition 6.2.1

In the case $B = 0$ condition (5.2.4) implies that $A + 1 = l(r)$, which is a linear equation with respect to r. We get (6.2.4) as soon as $A \neq 0$. If, instead, $B \neq 0$, then from definition of the EML we get the following quadratic equation with respect to $\bar{e} + r$

$$B\,(\bar{e} + r)^2 + A\,(\bar{e} + r)^2 + \bar{e} = 0 \quad . \tag{F.1.1}$$

The discriminant of this equation reads $D = A^2 - 4B\bar{e}$. Solving this equation in the case of positive discriminant one gets (6.2.5).

F.2 Proof of Proposition 6.2.2

The constant investment function has one or zero equilibria. For the increasing function consider the second inequality in (6.2.7). Substitution of the EML's slope (5.2.6) in equilibrium leads to

$$B\,(\bar{e} + r^*)^2 - \bar{e} < 0 \qquad \Leftrightarrow \qquad -A\,(\bar{e} + r^*) - 2\bar{e} < 0 \quad , \tag{F.2.1}$$

where we used the relation (F.1.1). Plugging corresponding equilibrium values from (6.2.5) and simplifying the resulting inequality, one gets

$$\sqrt{A^2 - 4B\bar{e}} + A < 0 \quad \text{in } r_1^* \qquad \text{and} \qquad \sqrt{A^2 - 4B\bar{e}} - A < 0 \quad \text{in } r_2^* \quad .$$

When $A > 0$, the left inequality is violated and therefore r_1^* is unstable as in the plot for region **IV** in Fig. 6.2. If $A < 0$ the right inequality is violated and r_2^* is unstable as in Fig. 6.2 for region **VII**.

Consider now the case of decreasing investment strategy $B < 0$. Then, as we showed in Section 6.2.1, the equilibria are such that $r_2^* < -\bar{e} < r_1^*$ (see also illustrations for regions **I** and **II** in Fig. 6.2). If the equilibrium return is negative, the first inequality in (6.2.7) leads to

$$B\,(\bar{e} + r^*)^2 - r^*\bar{e} > 0 \qquad \Leftrightarrow \qquad -A\,(\bar{e} + r^*) - \bar{e}(1 + r^*) > 0 \quad , \tag{F.2.2}$$

When $A \leq 0$, it, obviously, always holds with the opposite sign in feasible r_2^*, i.e. r_2^* is always unstable in this case. Analogously, when r_1^* is negative it will be unstable when $A > 0$.

Finally, let us consider the case when $A > 0$ and r_1^* is positive. The third inequality in (6.2.7) leads to

$$B\,(\bar{e} + r^*)^2\,(2 + r^*) + r^*\bar{e} > 0 \qquad \Leftrightarrow \qquad -A\,(\bar{e} + r^*)(2 + r^*) - 2\bar{e} > 0 \quad , \tag{F.2.3}$$

which always holds with the opposite sign. Thus, in all cases when two feasible equilibria exist one of them is unstable.

Appendix G

Proofs of Proposition in Chapter 7

G.1 Proof of Lemma 7.2.1

As we know $\rho_{t+1,n} = x_{t,n}\,(r_{t+1} + e_{t+1})$, so that:

$$
\begin{aligned}
\langle x_{t+1} \rangle_{t+1} &= \sum_n x_{t+1,n} \frac{w_{t+1,n}}{\sum_j w_{t+1,j}} = \sum_n x_{t+1,n} \frac{w_{t,n}\,(1+\rho_{t+1,n})}{\sum_j w_{t,j}\,(1+\rho_{t+1,j})} = \\
&= \frac{\sum_n x_{t+1,n}\, w_{t,n}\,(1 + x_{t,n}\,(r_{t+1}+e_{t+1}))}{\sum_n w_{t,n}\,(1 + x_{t,n}\,(r_{t+1}+e_{t+1}))} = \\
&= \frac{\sum_n x_{t+1,n}\, w_{t,n} + (r_{t+1}+e_{t+1}) \sum_n x_{t+1,n}\, x_{t,n}\, w_{t,n}}{\sum_n w_{t,n} \;+ (r_{t+1}+e_{t+1}) \sum_n x_{t,n} w_{t,n}}
\end{aligned}
$$

Dividing both numerator and denominator in the last expression on $\sum_n w_{t,n}$, one gets

$$
\langle x_{t+1} \rangle_{t+1} = \frac{\langle x_{t+1} \rangle_t + (r_{t+1}+e_{t+1})\, \langle x_{t+1}\, x_t \rangle_t}{1 + (r_{t+1}+e_{t+1})\, \langle x_t \rangle_t} \quad .
$$

Thus

$$
\langle x_{t+1} \rangle_{t+1} - \langle x_{t+1} \rangle_t = (r_{t+1}+e_{t+1}) \Big(\langle x_{t+1}\, x_t \rangle_t - \langle x_{t+1} \rangle_{t+1} \langle x_t \rangle_t \Big) \quad .
$$

From this equality the statement immediately follows.

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