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Asset Pricing Model with Heterogeneous Time Horizons

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Plan of the talk

- Capital Asset Pricing Model
- Agent-Based Models
- Investment Horizons
- The Model
- Conclusion
1. Capital Asset Pricing Model

What is it about?

Goal: to ”explain” the asset prices

Result: in equilibrium for any risky asset \( i \)

\[
E[\rho_i] - R = \beta_i (E[\rho_m] - R)
\]

where \( \rho_m \) is the return of market portfolio

Consequence: separation theorem

Consequence: all agents have the same market portfolio, i.e. the same share of any risky asset among other risky assets

Model is equilibrium, static and ”homogeneous”
1.1. Assumptions

Market

- asset market is complete and competitive
- assets are divisible
- unlimited short sales are allowed
- no taxes
- no transaction costs

Agents’ Expectations

- *homogeneous expectations*: all agents share a common probability distribution of returns for risky assets
- common investment planning horizon

Agents’ Behavior

- mean-variance utility functions express the preferences of investors
- agents hold the portfolio during one period

Remark

Equilibrium structure is changing with

- information
- investors’ preferences
- investors’ endowments
1.2. Intertemporal CAPM

Contributions

- Merton, 1973
- LeRoy, 1973
- Lucas, 1978

Features

- endogenous expectations
- \textit{price function}: asset price is a function of past realization of returns
- assumptions are even more restrictive

For example LeRoy assumes

- utility function with constant absolute risk aversion
- all investors are identical
- each investor can make an unbiased forecast of the price for the next period, in other words \textit{price functions derived under hypothesis of rational expectations}
1.3. Summary of theory evolution

- Mean-Variance Analysis: Markowitz (1956, 1959), Tobin (1958)
- Intertemporal CAPM: Merton (1973)
- Rational Expectations as a micro-foundation: LeRoy (1973), Lucas (1978)
- Agent-Based Models
2. Agent-Based Modeling

Driving forces (LeBaron, 2000)

- unsatisfactory assumptions of classical models
- empirical puzzles
  - equity premium puzzle
  - excess volatility
  - persistence in volatility
  - using of technical trading
- availability of financial data
- findings in experimental financial markets
Common ingredients

- *bottom-up* structure: from interaction of heterogeneous agents to properties of market dynamics

- both "fundamentalists" and "irrational" agents:
  - noise traders, chartists, ZI traders

- generation of non-trivial price dynamics: cyclical or chaotic

Examples

- simulations:
  - Gode and Sunder, 1993
  - Arthur et al., 1997
  - Chiaromonte, Dosi and Jonard, 1999
  - Levy, Levy and Solomon, 2000
  - Kirman and Teyssiere, 2000
  - Bottazzi, Dosi and Rebesco, 2002

- analytical studies:
  - Grossman and Stiglitz, 1980
  - De Long et al., 1991
  - Brock and Hommes, 1998
  - Chiarella and He, 2002
What we are interesting in here?

- what is the ”minimal” sufficient source of heterogeneity

- coexistence of heterogeneous traders in CAPM framework
  - agents can differ in predictors
  - agents can differ in preferences (for example, they might have different risk aversion coefficients)
  - agents can differ in investment horizons
  - agents can differ in wealth dynamics

- role of assumption about utility function

- role of market architecture
Key references


Differences among them

In Bottazzi (2002)

1. Agents make prediction about *returns*, not about prices

2. Agents are heterogeneous in the estimation of risk

3. Agents do not change own type
3. Investment Horizons

Hypothesis

The investment decision of a trader strongly depends on the time horizon on which his judgment about the profit and risk of a given investment is based.

Agent-based literature

- Agents are ”myopic”, i.e. they participate to the market in order to maximize the wealth at the next period.
- Do the results of models change if one introduces additional heterogeneity in terms of investment time horizons?
- Is it enough to introduce heterogeneity at the level of investment time horizons to break the predictions of standard representative-agent models?

Classical literature

- Merton, but there is no dynamics indeed...
4. The Model

4.1. Goals of the paper

- Search for the "minimal" heterogeneity to generate non-trivial price dynamics
- Qualitative consequences of the introduction of heterogeneity in time horizons

4.2. Conclusions

- Heterogeneity only in investment horizons is not enough
- In general, price dynamics is strongly affected by investment horizons
- Dynamics does depend on the way how agents estimate risk
- Overestimation of risk by fundamentalists leads to unstability of the system
4.3. Model setup

Bond $B$

- price 1
- constant interest rate $0 < R < 1$

Stock $A$

- price $p_t$ return $\rho_{t,t+\eta} = (p_{t+\eta} - p_t)/p_t$
- constant dividend $D$

Timing

- *beginning of period* $t$
- agent $i$ has $A_{i,t-1}, B_{i,t-1}$
- agent makes predictions $E^i_t[\rho_{t,t+\eta}]$ and $V^i_t[\rho_{t,t+\eta}]$
- agent maximizes expected utility $E^i_t[W_{i,t+\eta}] - \frac{\beta_i}{2} V^i_t[W_{i,t+\eta}]$
- the demand function of the agent $i$ $\Delta A_{i,t}(p) = -A_{i,t-1} + \tilde{A}_{i,t}(p)$
- the market clearing condition is $\sum_{i=1}^{N} \Delta A_{i,t}(p) = 0$
- solution is price $p_t$
- now agent $i$ has $A_{i,t}, B_{i,t}$
- dividend $D$ and interest rate $R$ are paid
- *end of period* $t$
4.4. Modeling different time horizons

Agent with a time horizon $\eta > 0$ periods

1. – does not correct the portfolio between time $t$ and $t + \eta$

2. – participates in the market at each period
   – continuously corrects the portfolio composition
   – takes into account the fact that the portfolio will be revised each period

3. – participates in the market at each period
   – continuously corrects the portfolio composition
   – does not take into account the fact that the portfolio will be revised
   – so he maximizes at period $t$ his expected wealth at period $t + \eta$

$$W_{t+\eta} = (1 - x_t)W_t(1 + R)^\eta + x_t W_t \left(1 + \rho_{t,t+\eta} + \frac{\bar{p}((1 + R)^\eta - 1)}{p_t}\right)$$

$x_t$ share of wealth invested in the risky asset

$\bar{p} = D/R$ fundamental price
Thus

- demand for the stock

\[
\tilde{A}_{i,t}(p) = \frac{E_t^i[\rho_{t,t+\eta_i}] + ((1 + R)^{\eta_i} - 1)(\bar{p} - 1)}{\beta_i p \bar{V}_t^i[\rho_{t,t+\eta_i}]}
\]

- the market clearing condition

\[
\sum_{i=1}^{N} \Delta A_{i,t}(p) = 0
\]

- the market clearing condition

\[
\bar{A} = \langle \tilde{A}_{i,t}(p) \rangle_i \quad \bar{A} = A_{TOT}/N
\]

- new price of the risky asset \( p_t \) is a solution of

\[
p^2 \bar{A} - p \left\langle \frac{E_t^i[\rho_{t,t+\eta_i}] - \hat{R}_i}{\beta_i V_t^i[\rho_{t,t+\eta_i}]} \right\rangle_i - \left\langle \frac{\bar{p}\hat{R}_i}{\beta_i V_t^i[\rho_{t,t+\eta_i}]} \right\rangle_i = 0
\]

where \( \hat{R}_i = (1 + R)^{\eta_i} - 1 > 0 \)

- non-linear pricing equation (cf. Brock-Hommes model)

- unique positive root

- in the situation of complete certainty, the only solution is \( \bar{p} \)
4.5. Two classes of agents

**Chartists** behave as econometricians

- $\lambda \in [0, 1)$ – length of the agent’s memory
- one period forecast
  \[
  E^c_t[\rho_{t,t+1}] = R^\text{MA}_{t-1} = (1 - \lambda) \sum_{\tau=2}^{\infty} \lambda^{\tau-2} \rho_{t-\tau}
  \]
  \[
  V^c_t[\rho_{t,t+1}] = V^\text{MA}_{t-1} = (1 - \lambda) \sum_{\tau=2}^{\infty} \lambda^{\tau-2} [\rho_{t-\tau} - R^\text{MA}_{t-1}]^2
  \]
- recursive relation
  \[
  R^\text{MA}_{t-1} = \lambda R^\text{MA}_{t-2} + (1 - \lambda) \rho_{t-2}
  \]
  \[
  V^\text{MA}_{t-1} = \lambda V^\text{MA}_{t-2} + \lambda (1 - \lambda) (\rho_{t-2} - R^\text{MA}_{t-2})^2
  \]
- long-term forecast
  \[
  E^c_t[\rho_{t,t+\eta}] = \eta E^c_t[\rho_{t,t+1}] = \eta R^\text{MA}_{t-1}
  \]
  \[
  V^c_t[\rho_{t,t+\eta}] = \eta V^c_t[\rho_{t,t+1}] = \eta V^\text{MA}_{t-1}
  \]
Fundamentalists believe in mean-reverting dynamics

- $\theta \in [0, 1]$ – perception of the reactivity of market
- one period price forecast
  \[ E_t^f[p_{t+1}] = p_t + \theta(\bar{p} - p_t) \]
- long-term return forecast
  \[ E_t^f[\rho_{t,t+\eta}] = \left(1 - (1 - \theta)^\eta\right)\left(\frac{\bar{p}}{p_t} - 1\right) \]
- long-term expected variance
  \[ V_t^f[\rho_{t,t+\eta}] = \eta V_t^f[\rho_{t,t+1}] = \eta V_{t-1}^{MA} \]
4.6. **Fundamentalists vs. Chartists I**

- \( f_1 \) and \( f_2 \) are the fractions of fundamentalists and chartists
- all investors have the same time horizon 1
- the market clearing condition

\[
\beta \tilde{A} p^2 V_{t-1}^{MA} + p \left( R + f_1 \theta - f_2 R_{t-1}^{MA} \right) - \bar{p} \left( R + f_1 \theta \right) = 0
\]

- parameters and notation:
  \[
  \gamma = \frac{\beta \tilde{A}}{f_2} \quad x(t) = \gamma p_t \\
  r = \frac{(R + f_1 \theta)}{f_2} \quad y(t) = R_t^{MA} \\
  d = \frac{r \bar{p} = (D + f_1 \theta \bar{p})}{f_2} \quad z(t) = V_t^{MA} \\
  s = d \gamma
  \]

- 3-dimensional system

\[
\begin{align*}
  x(t+1) &= \left( y(t) - r + \sqrt{(y(t) - r)^2 + 4sz(t)} \right)/2z(t) \\
  y(t+1) &= \lambda y(t) + (1 - \lambda) \left( \frac{g(y(t),z(t))}{x(t)} - 1 \right) \\
  z(t+1) &= \lambda z(t) + \lambda (1 - \lambda) \left( \frac{g(y(t),z(t))}{x(t)} - 1 - y(t) \right)^2
\end{align*}
\]
4.7. The local stability of the system

- system has only one fixed point \((\gamma \bar{p}, 0, 0)\).

- necessary and sufficient condition for the local stability of this point:
  \[
  r + \lambda > 1 \quad \quad \quad \quad \quad r = (R + f_1 \theta)/f_2
  \]

- system has Hopf bifurcation
4.8. Fundamentalists vs. Chartists II

- $f_1$ and $f_2$ are the fractions of fundamentalists and chartists
- $\eta_1$ and $\eta_2$ are the time horizons of fundamentalists and chartists
- parameter
  \[
  r = \frac{1}{\eta_1} \frac{f_1}{f_2} (\theta + R) B_{\eta_1}(R, \theta) + \frac{1}{\eta_2} R B_{\eta_2}(R, 0)
  \]

with $\eta$ increasing the function increases exponentially to the infinity
5. Discussion

- convergence to the fixed point comes with
  - increasing of the length of the agents’ memory $\lambda$
  - increasing of the share of fundamentalists on the market $f_1$
  - increasing of the risk-less return $R$
  - increasing of the perception of fundamentalists about the efficiency of the market $\theta$
• price dynamics is strongly affected by investment horizons

Parameters are $R = 0.05$, $D = 1.0$, $\bar{A} = 2$, $\beta = 1$, $\lambda = 0.9$, $f_1 = 0.2$, $\theta = 0.4$, $\eta_2 = 1$. The initial conditions are $p = 1$, $R^{MA} = 0.01$, $V^{MA} = 0.0001$. 
- effect of increasing of time horizons $\eta_1$ and $\eta_2$ is asymmetric

Parameter $r$ as a function of $\eta_1$ and $\eta_2$. The values of other parameters are $R = 0.05$, $f_1 = 0.2$, $\theta = 0.3$. The curve on the horizontal space confines the closed region in space $(\eta_1, \eta_2)$ where the fundamental price is unstable fixed point.
• the moderate time horizons of fundamentalists can bring the system to chaotic behavior.

Bifurcation diagram. The price support of a 1000 steps orbit (after a 1000 steps transient) is shown for 50 distinct values of $\eta_1$ from 1 to 50.
6. Conclusion

- Heterogeneity only in investment horizons is not enough
- In general, price dynamics is strongly affected by investment horizons
- Different types of population contribute in different way
- Dynamics does depend on the way how agents estimate risk
- Overestimation of risk by fundamentalists leads to unstability of the system