Exercise 1

Since we know that the decision maker prefers the lottery that maximizes her expected utility, we have to compute the expected utility of the two lotteries. The lottery with the highest expected utility is the one that the decision maker will prefer. For each lottery, the computation of the expected utility E is simply obtained with a sum over the possible outcomes

$$E = \sum_i u(x_i) p_i$$

where u is the decision makers' utility function, which describes her preferences; $u(x_i)$ is the utility associated with the money payed by the i-th outcome x_i; and p_i is the probability of occurence of the i-th outcome. Obviously $\sum_i p_1 = 1$.

In our case the expected utility of lottery one is

$$E_1 = 0.06\sqrt{100} + (1 - 0.06)\sqrt{0} = 0.6$$

while the expected utility of lottery two is

$$E_2 = 0.03\sqrt{200} + (1 - 0.03)\sqrt{0} \sim 0.42 \ .$$

Since $E_1 > E_2$, we can conclude that the decision maker prefers lottery one.

Exercise 2

We have to assign an utility level to each action, $A1$ and $A2$, conditional on each outcome, $O1$ and $O2$. So we have a total of four utility levels: $u(A1, O1)$, $u(A1, O2)$, $u(A1, O2)$ and $u(A2, O2)$. We use the available information to rank the utility of the different actions. Let see how to do it considering the piece of information one at a time:

- If the market shrinks (O1) the best thing to do is not to invest (A2): this means that $u(A2, O1) > u(A1, O1)$.
- If the market increases or remains the same (O2) the best thing to do is to invest (A1): this means that $u(A1, O2) > u(A2, O2)$.
- If the investment is performed (A1) the best outcome is outcome O2: this means that $u(A1, O2) > u(A2, O1)$.
- If the investment is not performed (A2) the best outcome is O1: this means that $u(A2, O1) > u(A2, O2)$.
- Investing in market expansion or stationary state is better than non investing in market contraction: this means that $u(A1, O2) > u(A2, O1)$.
- Investing in market contraction is equivalent to non investing in market expansion or stationary state: this means that $u(A1, O1) = u(A2, O2)$.

Using the relations above we can order all the utility levels writing

$$u(A1, O2) > u(A2, O1) > u(A1, O1) = u(A2, O2) \ .$$

Since the expected utility theory implies that the preference structure is invariant by linear transformation of utilities, we can always fix two utility levels. Let assume that $u(A1, O1) =$

$u(A2,O2) = 0$ and $u(A1,O2) = 1$ and the only utility level that remain to be determined is $u = u(A2,O1)$ with $u \in (0,1)$. For this purpose we use that fact that "if the decision maker believes that the probability of outcome O1 is .8, then she is indifferent between investing and not-investing". This means that if the probability of outcome $O1$ is .8, the expected utility of the two actions, investing or not investing, is the same. The expected utility of action 1 is

$$U(A1) = U(A1,O1)0.8 + U(A1,O2)0.2 = 0.2 \; ,$$

where I have substituted the previously fixed utility levels. The expected utility of action 2 is

$$U(A2) = U(A2,O1)0.8 + U(A2,O2)0.2 = 0.8u \; .$$

By equating the two previous expressions

$$0.2 = 0.8u$$

one has $u = 0.25$ and the preference structure of the investor is completely defined.

With these preferences, if the probability of $O1$ is presumed equal to .4, one has

$$U(A1) = U(A1,O1)0.4 + U(A1,O2)0.6 = 0.6$$

and

$$U(A2) = U(A2,O1)0.4 + U(A2,O2)0.6 = 0.1 \; ,$$

so that $U(A1) > U(A2)$ and the investor will chose action $A1$.

Exercise 3

The first part of the problem is easily solved by taking the first and second derivatives of the utility function

$$u_a^{'}(x) = \frac{a}{(x+a)^2} \; , \quad u_a^{''}(x) = -\frac{2a}{(x+a)^3} \; .$$

Since on the entire support $u_a^{'}(x) > 0$ and $u_a^{''}(x) < 0$ the function is increasing and concave. Thus, it represents risk-averse preferences. The Arrow-Pratt coefficient of absolute risk aversion is

$$A_a(x) = -\frac{u_a^{''}(x)}{u_a^{'}(x)} = \frac{2}{x+a} \; .$$

Notice that the derivative of A with respect to the parameter a,

$$\frac{d}{da}A_a(x) = -\frac{2}{(x+a)^2} \; ,$$

is negative, so the risk aversion is decreasing with the increase of a. As a consequence $u_{10}(x)$ is more risk averse than $u_{20}(x)$.

For the second part of the problem, we have to write down the expected utility of a generic portfolio. We know that the argument of the utility function is going to be the wealth in the next period. Assuming that in period one a wealth w_1 is invested in the first asset and a wealth w_2 is invested in the second, given the market structure (that is the rules governing the payoff of the two securities) the investor will receive an amount equal to $1.2w_1 + 2.4w_2$ if the risky asset pays its yield or equal to $1.2w_1$ if the risky asset pays nothing. The first outcome occurs with probability 0.8, the second with probability 0.2. Thus the expected utility reads

$$0.8\, u(1.2w_1 + 2.4w_2) + 0.2\, u(1.2w_1) \, .$$

Substituting the explicit expression for the utility function, one has

$$U(w_1, w_2) = 0.8\frac{1.2w_1 + 2.4w_2}{1.2w_1 + 2.4w_2 + a} + 0.2\frac{1.2w_1}{1.2w_1 + a}$$

The problem of the investor is to maximize the expression above under the budget constraint in period one, that is $w_1 + w_2 = 3000$:

$$(w_1^*, w_2^*) = \text{argmax}\, U(w_1, w_2) \quad \text{s.t.} \quad w_1 + w_2 = 3000 \ .$$

The easiest way to proceed is to directly substitute the constraint in the equation. Let $x = w_2$ be the wealth invested in the risky security so that $w_1 = w - x$, the problem becomes the maximization in x of

$$U(x) = 0.8\frac{1.2w + 1.2x}{1.2w + 1.2x + a} + 0.2\frac{1.2w - 1.2x}{1.2w - 1.2x + a}$$

with $x \in [0, 1]$ and where $w = 3000$ denotes the initial endowment. The first order condition (f.o.c) reads

$$0.8\frac{1.2a}{(1.2w + 1.2x + a)^2} - 0.2\frac{1.2a}{(1.2w - 1.2x + a)^2} = 0$$

which after a little algebra reduces to

$$x = \frac{w}{3} + \frac{a}{3r_1} \ .$$

Substituting $w = 3000$, one has that the quantity invested in the risky security for $u_{36}(x)$ is 1010 while it is 1005 for $u_{18}(x)$.

Giulio Bottazzi, Pisa 06/06/2011