Solution of the exam of "Decision under uncertainty", Pisa 27/06/2011

Exercise 1

The first derivative of $u(x)$ reads

$$u'(x) = 1 + \frac{1}{x^2} \ .$$

It is strictly positive, so that $u(x)$ is strictly increasing on its entire domain of definition. Conversely, the second derivative is negative

$$u''(x) = -\frac{2}{x^3}$$

which implies that $u(x)$ is concave.

Consider an endowment of 4. The decision maker uses $u(x)$ to evaluate amounts of money in period 2. If the endowment is retained the utility it gives in period 2 is

$$u(4) = 4 - \frac{1}{4} = \frac{15}{16} \ .$$

If instead the initial amount is invested in lottery L_2, it generates an expected utility equal to

$$\frac{1}{4}u(16) + \frac{3}{4}u(1) = \frac{1}{16}\left(64 - \frac{1}{4}\right) = \frac{63.75}{16}$$

Since $63.75 > 15$, it is clear that the decision maker will invest her endowment in lottery L_2. Notice that the exercise did not ask about the fraction of the initial endowment optimally invested in lottery L_2. Only two options were available to the decision maker: to invest all the endowment or to invest nothing.

Next we have to find the endowment x for which the decision maker is indifferent between keeping the amount x for period 2 or investing it entirely in lottery L_1. That is, we have to find the amount of money x having an utility equal to the utility expected from investing the same amount in lottery L_1. In other terms we have to solve the following

$$u(x) = \frac{1}{3}u(3x) + \frac{2}{3}u(x/3) \ ,$$

which substituting the expression for u becomes

$$x - \frac{1}{x} = \frac{1}{3}\left(3x - \frac{1}{3x}\right) + \frac{2}{3}\left(\frac{x}{3} - \frac{3}{x}\right) \ .$$

After obvious simplifications and collecting similar terms one has

$$\frac{1}{x}\frac{10}{9} = x\frac{2}{9}$$

which, multiplying both sides by x, reduces to

$$x^2 = 5 \ .$$

The amount of money we where looking for is then $x = \sqrt{5}$.

Finally, we are looking for the amount of money y whose utility is equal to the utility of a lottery paying 5 with probability 1/2 and 1 otherwise. In other terms we are interested in the solution of the following equation

$$u(y) = \frac{1}{2}u(5) + \frac{1}{2}u(1)$$

which substituting the expression for u becomes

$$y - \frac{1}{y} = \frac{12}{5}; .$$

Multiplying both sides by y this reduces to the second order equation

$$y^2 - \frac{12}{5}y - 1 = 0 .$$

Applying the well known formula, the solutions of this equations read

$$y = \frac{6}{5} \pm \sqrt{\left(\frac{6}{5}\right)^2 + 1} .$$

Clearly only the positive solution is acceptable (no negative amount of money are allowed) so that finally one has

$$y = \frac{6 + \sqrt{61}}{5} .$$

Exercise 2

We are looking for four utility levels, namely $u(A,s)$, $u(B,s)$, $u(A,f)$ and $u(B,f)$, labeled according to the firm potentially chosen for the investment A or B and the outcome of the innovation, a success (s) or a failure (f), We start by expressing the formal implications of the provided information

- for both firms, if the scientific innovation is successful, then the generated revenues are higher: this implies that $u(A,s) > u(A,f)$ and $u(B,s) > u(B,f)$.
- if the scientific innovation does not have success, firm A will generate higher revenues than firm B: this implies that $u(A,f) > u(B,f)$
- if the scientific innovation is successful, revenues of firm A will double: since the utility function is linear, this implies that $u(A,s) = 2u(A,f)$
- if the probability of success of the scientific innovation is $\pi = 1/10$, then investing in the two firms is equivalent: equating the expected utility of the two potential investments, this implies that
$$\frac{1}{10}u(A,s) + \frac{9}{10}u(A,f) = \frac{1}{10}u(B,s) + \frac{9}{10}u(B,f) .$$

From the first two conditions it is clear that $u(B,f)$ must be the lowest utility. Since we are always allowed a translation in utility levels, we set $u(B,f) = 0$. Then substituting the third condition in the last we have

$$\frac{11}{10}u(A,f) = \frac{1}{10}u(B,s)$$

which gives $u(A, f) = u(B, s)/11$ and $u(A, s) = 2u(B, s)/11$. Setting $u(B, s) = 1$, because we are always allowed for a rescaling of the utility levels, gives the final set of utilities.

With these utility levels, let's compute the expected utility of the investment in firm A and B, if the probability of success of the innovation is $1/5$. Investing in A will give and expected utility of

$$\frac{1}{5}u(A, s) + \frac{4}{5}u(A, f) = \frac{6}{55} \ .$$

Alternatively, investing in B will give and expected utility of

$$\frac{1}{5}u(B, s) + \frac{4}{5}u(B, f) = \frac{1}{5} \ .$$

Since $1/5 > 6/55$ firm B will be selected.