Solution of the exam of "Decision under uncertainty", Pisa 07/06/2011

Exercise 1

The increasing and concave nature of the function $u_a(x)$ is directly proven by taking the first and second derivatives

$$\frac{d}{dx}u_a'(x) = \frac{1}{2\sqrt{a+x}} > 0 \quad \frac{d}{dx}u_a''(x) = -\frac{1}{4(a+x)^{3/2}} < 0 \ .$$

Then from these expressions compute the absolute risk aversion coefficient

$$r(a,x) = -\frac{u_a''(x)}{u_a'(x)} = \frac{1}{2(a+x)}$$

For any x, the risk aversion coefficient of the function $u_a(x)$ is decreasing with a, indeed

$$\frac{d}{da}r(a,x) = -\frac{1}{2(a+x)^2} < 0 \ .$$

This implies that $u_{10}(x)$ is more risk averse that $u_{20}(x)$. Notice that this conclusion can be reached because $r(a,x)$ decreases with a for *any* x.

The expected utility of lotteries L_1 and L_2 read

$$U_a(L_1) = \pi\sqrt{a+A} + (1-\pi)\sqrt{a}$$

and

$$U_a(L_2) = \frac{\pi}{2}\sqrt{a+4A} + (1-\frac{\pi}{2})\sqrt{a}$$

respectively. By equating the two expressions and simplifying one gets

$$2\sqrt{a+A} = \sqrt{a+4A} + \sqrt{a}$$

which after taking the square of the two sides and simplifying becomes

$$a = \sqrt{(a+4A)a}$$

and by squaring again one finally gets

$$aA = 0 \ .$$

Since by assumption $A > 0$, the unique solution of the problem is $a = 0$.

Denote with L^* the lottery paying $2A$ with probability $\pi/2$ and zero otherwise. Its expected utility reads

$$U_a(L^*) = \frac{\pi}{2}\sqrt{a+2A} + (1-\frac{\pi}{2})\sqrt{a} \ .$$

If there exists an a for which L_3 is preferred to L_1, for that a it should be

$$U_a(L^*) > U_a(L_1) \ .$$

Substituting the expressions of the utility functions the inequality reduces to

$$2\sqrt{a+A} < \sqrt{a+2A} + \sqrt{a}$$

Taking the square of both sides, simplifying and taking the square again leads to the inequality

$$A^2 < 0$$

which is never satisfied. Then we can conclude that there are no values of a such that L_3 is preferred to L_1.

The certainty equivalent of L_3 is defined as the amount of money c such that

$$u_a(c) = \sqrt{a+c} = \int dF(x) u_a(x) = \int dF(x) \sqrt{a+x} \, .$$

where F is the probability distribution associated to L_3. Substituting the expression for F in the right-hand side one has

$$\frac{1}{3}\int_0^3 \sqrt{a+x} = \frac{2}{9}[(a+x)^{3/2}]_{x=0}^{x=3} = \frac{2}{9}\left((a+3)^{3/2} - a^{3/2}\right)$$

so that

$$c = \frac{4}{81}\left((a+3)^{3/2} - a^{3/2}\right)^2 - a$$

By setting $a = 1$ one gets $c \sim 1.4$. As expected, since the rule is risk averse, the certainty equivalent is lower than 1.5, the expected payoff of the lottery. *Since the risk aversion is decreasing in a, the certainty equivalent of L_3 should decrease when $a \to \infty$. Can you find its limit?*

Exercise 2

If the decision maker's preferences are consistent with the Expected Utility Theory, then they satisfy the axiom of independence. That is, given two lotteries L_1 and L_2, if $L_2 \succ L_1$ one has

$$\alpha L_2 + (1-\alpha) L_3 \succ \alpha L_1 + (1-\alpha) L_3$$

for *any* lottery L_3 and for *any* $\alpha \in [0, 1]$.

Consider the large outcome space that contains all the possible outcomes, namely $\{0, 8, 10, 12\}$. On this space the lotteries in the exercise have the following representation (each row list the probabilities associated to a different lottery)

	0	8	10	12
L_1	0.5	0	0.5	0
L_2	0.4	0.6	0	0
L_3	0.6	0	0	0.4
L'_1	0.55	0	0.25	0.2
L'_2	0.5	0.3	0	0.2

Table 1: Lotteries on the enlarged outcome space

Inspecting this table it is immediate to see that $L'_1 = 1/2L_1 + 1/2L_3$ and $L'_2 = 1/2L_2 + 1/2L_3$. Thus the proof immediately follows from the axiom of independence.