Exam of "Mathematics for Economists", Pisa 26/02/2014

Solve the following exercises, motivating your answers. Please notice that: you can keep with you the notes and the textbook of the course, but no other books are allowed; you can use a pocket/scientific calculator, but any device able to perform symbolic computation is not allowed; at the end of the exam, you are required to deliver all the paper (draft included) to the supervisor.

Exercise 1

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{O} \subset \mathbb{N}$ the set of odd numbers. Prove that the set $B = \{\{n, n+1\} | n \in \mathbb{O}\} \subseteq 2^{\mathbb{N}}$ is the base of a topology. Let $X = (\mathbb{N}, T)$ be the topological space generated by B. In X consider the set $A = \{2, 3, 4\}$ and find its interior, its boundary and its derivative set.

Exercise 2

Consider the sequence defined by recursion

$$\begin{cases} a_1 & = \sqrt{2} \\ a_{n+1} & = \sqrt{2 + a_n} \end{cases} .$$

Prove that the sequence converges and compute its limit. *Hint: Remember what you know about increasing sequences bounded from above.*

Exercise 3

Consider the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{2^n + 1} .$$

Prove that the series is convergent for $x = 1$ and divergent for $x = 2$. Compute its radius of convergence.

Exercise 4

Let a and b be two positive real numbers lower than 1, with $a < b$. Find the measure $\alpha(x)$ such that for any function f continous in $[0, 1]$ it is

$$\int_0^1 d\alpha(x)\, f(x) = f(a) + f(b) + \int_a^b dx f(x) .$$

Is the measure $\alpha(x)$ unique? Can the property above be extended to all Riemann integrable functions in $[0, 1]$?

Exercise 5

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = x^2 \int_0^y dt\, g(t)$$

where $g(t)$ is a continuous function on $\mathbb{R}$. Find the domain of definition of $f(x, y)$. Is the function $f(x, y)$ continuous? Is it differentiable? Prove that $(0, 0)$ is a critical points of $f(x, y)$. What assumptions on g are necessary for $(0, 0)$ to be a local minimum?