Microeconomics

Angelo Secchi

PSME
Academic Year 2011/12

Who are we?

- Angelo Secchi
  - Université Paris 1 Panthéon-Sorbonne, Centre d'Economie
  - Bocconi University (undergrad. economics), Sant'Anna School of Advanced Studies (PhD)
  - industrial organization, industrial dynamics and applied econometrics
- Senne Jung
  - PhD candidate Paris School of Economics and Université Paris 1 Panthéon-Sorbonne
  - Yonsei University (undergrad. mathematics), Seoul National University (MA in Economics), Paris school of economics(MA in Economics)
  - applied microeconomics, labour economics, behavioral economics

Organization

Everything Microeconomics EPI web page

- Readings
  - no required textbook
  - Jehle and Reny - Advanced Microeconomic Theory (expensive)
  - Varian - Microeconomic Analyses
  - further readings and Syllabus on EPI
  - slides available Monday evenings on EPI, print them out before classes
  - warnings on slides: howto use them, handle them with care

- Contacts
  - angelo.secchi@univ-paris1.fr Put [MICRO] in the Subject
  - in case you need to talk with me, drop an email to make an appointment

Organization - cont'ed

- Exam: written, closed book
  - MIDTERM November 3,2011 2pm-5pm
  - FINAL January 10,2012 2pm-5pm
- Grading policy
  - 50% midterm and 50% final
  - grades range: $0 - 20$. There is no formal Failure threshold.
  - To validate the semester you must have an average grade of 10. If your semester average is lower than 10, you must retake all the exams with a grade lower than 10.
  - To validate the year and henece to obtain the Master you must have an overall average grade of 10.
- Code of Conduct: honesty, integrity and respect.
  - ignorance is not an excuse, check the Code on the EPI
  - Board of Examiners is very strict in applying the Code

Organization - cont'ed

- Class timing(tentative)

2.00-3.30	3.30-3.45	3.45-4.45	4.45-5.00	
class	s break	class	R&S	Q&A

- Maths

  Maths is not the subject of this course but the formalization of economic intuitions discussed in ECON101 is the main aim of our classes. If you have problems contact one of your tutors.

- (Hopefully useless) Notes
  - Exams will not be rescheduled. No exceptions. Do not make plans or travel reservations for the exam week. This is not a valid excuse for rescheduling exams.
  - In case of exceptional events contact, as soon as possible, one of your tutors.
  - Please turn off cell phones and other noisemakers during class.

Tentative schedule

20	Sep	Introduction.
27	Sep	Consumer theory.
4	Oct	Duality, Slutsky and the integrability problem.
11	Oct	Production theory.
18	Oct	Uncertainty.
25	Oct	Partial equilibrium analysis. Catch up and review before the midterm exam.
3	Nov	MIDTERM EXAM (covers classes until 18 Oct)
8	Nov	General equilibrium in pure exchange economy.
15	Nov	General equilibrium with production. Core.
22	Nov	Game theory I: strategic form games.
29	Nov	Game theory II: extensive form games.
6	Dec	Adverse selection, moral hazard and the agency problem.
13	Dec	Catch up and review before the final exam.

Ariel Rubinstein's Warnings

Many economists have strong and conflicting views about what economic theory is

- set of theories that can/should be tested
- bag of tools to be used by economic agents
- framework through which professional and academic economists view the world

Ariel Rubinstein's Warnings

Disappointment for those of you who have come to this course with practical motivations

- economic theory is no more than an arena for the investigation of concepts we use in thinking about economics in real life and models provide a language to try to understand reality better BUT
- I do not view economic models as an attempt to describe the world or to provide tools for predicting the future
- I object to looking for an ultimate truth in economic theory
- I do not expect it to be the foundation for any policy recommendation

However, an economic model differs substantially from a purely mathematical model in that it is a combination of a mathematical model and its interpretation.

Microeconomics

Microeconomics is a collection of models in which the primitives are details about the behavior of units called economic agents (not necessarily individuals).

An economic agent is described in our models as a unit, that following a deliberation process(rational choice), make a choice from a set of available alternatives. The process consists in

- asking himself “What is desirable?”
- asking himself “What is feasible?”
- choosing the most desirable from among the feasible alternatives.

Note the order of the stages: rational economic agent’s desires are independent of the set of alternatives. Also rationality in economics does not contain judgments about desires.

Economists have become increasingly aware that almost all people, almost all the time, do not practice this kind of deliberation!

Is it meaningful to talk about the concept of “being good” even in a society where all people are evil? Similarly . . .

Further reading

Rubinstein, A., “Dilemmas of an economic theorist”
http://arielrubinstein.tau.ac.il/papers/74.pdf[clickable]

Preferences

Why do we start with preferences?

"Preferences" are one of the building block of microeconomic models

- imagine you want to fully describe the preferences of an economic agent over a set X of alternatives
- what should this description include?
- is there any requirement this system of preferences should fulfill?

Let's introduce this concept in a simple intuitive manner with a modeling exercise.

Preferences - Questionnaire Q

- consider an economic agent and a set of alternatives X
- for all x and y in X the agent is confronted with the question $Q(x, y)$: how do you compare x and y?
- select one of the following
  - ☐ I prefer x to y, $x \succ y$
  - ☐ I prefer y to x, $x \prec y$
  - ☐ I'm indifferent, $x \sim y$

Preferences - Legal answer

- one and only one choice is permitted
- we exclude answers such as: they are incomparable, I do not know, I prefer x to y but also y to x, it depends on my girlfriend's taste or on the weather
- we ignore the intensity of preferences

Do we qualify all the legal answers to the questionnaire Q qualify as preferences over the set X? NO

Preferences - Consistency restrictions

We adopt two "consistency" restrictions

- $Q(x, y)$ must be equal to $Q(y, x)$ [No order effect]

- the answers to $Q(x, y)$, $Q(y, z)$ and $Q(x, z)$ must be consistent. That is, if the answer to $Q(x, y)$, $Q(y, z)$ is "preference" then also $Q(x, z)$ must be answered "preference". Same in case of indifference. [Transitive]

Preferences - Questionnaire R

A second way to think about preferences is via a different questionnaire. Again

- consider an economic agent and a set of alternatives X
- for all x and y in X the agent is confronted with the question $R(x, y)$: is x at least as preferred as y?
- select one of the following
  - ☐ Yes
  - ☐ No

Preferences - Legal answer

- one and only one choice is permitted
- $R(x, y)$ must be equal to $R(y, x)$ [No order effect]
- if the answers to $R(x, y)$ and $R(y, z)$ is "yes" then also the answer to $R(x, z)$ must be "yes" [Transitive]

The two representations of agent's preferences are equivalent

Preferences - caveats and limitations

Transitivity seems a reasonable assumption. However,

- real experiments show frequent violations of transitivity
- it excludes individuals that base their judgment on procedures such
  - aggregation of primitive considerations
  - similarity among alternatives

Preferences - Aggregation and intransitivity

Aggregation of primitive considerations may break transitivity

Suppose an individual has to choose among 3 alternatives on which he has three primitive considerations and finds an alternative better than another one if the majority of primitives support it.

Consider the following rankings table

PRIMITIVES	x	y	z
consid. A	1	2	3
consid. B	3	1	2
consid. C	2	3	1

In this case $x \succ y$, $y \succ z$ but $z \succ x$

Preferences - Similarities and intransitivity

Similarity among alternatives may break transitivity

Consider an individual whose attitude is “the larger the better”. However he cannot distinguish among alternatives unless their difference is greater than 1.

$$
\begin{aligned}
f(x,y) &= x \quad \textit{if} \quad x \geq y+1 \\
f(x,y) &= I \quad \textit{if} \quad |x-y| < 1
\end{aligned}
$$

then $1.5 \sim 0.8$, $0.8 \sim 0.3$ but $1.5 \sim 0.3$ is not true

Utility

Think of examples of preferences.

- if the alternatives are few we may simply rank them:
  base additive colors(RGB): red$\succ$green$\succ$blue
- if alternatives are many (most often)
  basketball players: I prefer the taller basket players

All these examples can naturally be specified by a statement of the form

$$\mathrm{x} \succ \mathrm{y} \text{ if } \mathrm{V}\ (\mathrm{x}) \geq \mathrm{V}\ (\mathrm{y})$$

where $V : X \to \mathbb{R}$ is a function that attaches a real number to each element in the set of alternatives X. V(x) is called utility function and $\succeq$ is said to have an utility representation

Utility

Comments

- Note that the statement x$\succ$y if V (x)$\geq$ V (y) always defines a preference relation since the relation $\geq$ on $\mathbb{R}$ satisfies completeness and transitivity

- It is possible to avoid the notion of a utility representation and to do economics only with the notion of preferences

- when defining a preference relation using a utility function, the function has an intuitive meaning that carries with it additional information

- in contrast, when the utility function is formed to represent an existing preference relation, the utility function per se has no meaning.

Utility

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- It is possible to avoid the notion of a utility representation and to do economics only with the notion of preferences
- when defining a preference relation using a utility function, the function has an intuitive meaning that carries with it additional information
- in contrast, when the utility function is formed to represent an existing preference relation, the utility function per se has no meaning. Absolute values are meaningless!

Choice

- until now we have avoided any reference to agent's behavior, we only talked about preferences as a summary of the decision maker's mental attitude toward a set of alternatives
- consider now the usual set of alternatives X, a choice problem may be viewed as a nonempty subset of X, and we refer to a choice from $A \subseteq X$ as specifying one of A's members
- representing a choice problem in this way implies that agent's choice does not depend on the way alternatives are presented
  - ignores the order in which alternatives are presented
  - ignores the number of time alternatives appear in A
  - ignores if the are "defaults"

Choice - Rationality assumption

It is typically assumed in economics that choice is an outcome of rational deliberation

economic agent has in mind a preference relation $\succeq$ on the set X and, given any choice problem $A \subseteq X$, he chooses $x \in A$ OPTIMAL with respect to $\succeq$

Choice - Critiques to rationality

- Economists were often criticized because of the assumption of rationality
- economists' defense: we don't really need this assumption. All we need to assume is that the decision maker's behavior can be described as if he were maximizing some preference relation
- a modern attack, behavioral economics. Economics+Psychology, A. Tversky and D. Kahneman (Nobel laureate in economics in 2002). They have demonstrated not only that rationality is often violated, but that there are systematic reasons for the violations...

Choice - Critiques to rationality

FRAMING

Consider the following formulations of a plan to react to a national pandemia

- formulation A
  - (a) 400 people will die
  - (b) 0 will die with probability 1/3 and 600 will die with probability 2/3

- formulation B
  - (c) 200 people will be saved
  - (d) 600 will be saved with probability 1/3 and 0 will be save with probability 2/3

Choice - Critiques to rationality

The two questions were presented to 2 groups of people: 78% chose b and only 28% chose d

Both questions presented in the above order to game theory students: 78% chose b and only 49% chose d.

- large proportion of subjects gave different answers to the two problems
- results highlight the sensitivity of choices to the framing of alternatives
- what is more damaging for rational decision making than observing different answer to the same question only because they are presented differently?

Choice - Critiques to rationality

SIMILARITIES

Consider the following gambling games

(a)

color	white	red	green	yellow
probability	90	6	1	3
price €	0	45	30	-15

(b)

color	white	red	green	yellow
probability	90	7	1	2
price €	0	45	-10	-15

Which one would you choose?

Choice - Critiques to rationality

Consider now the following gambling games

	color	white	red	green	blue	yellow
(c)	probability	90	6	1	1	2
	price €	0	45	30	-15	-15

	color	white	red	green	blue	yellow
(d)	probability	90	6	1	1	2
	price €	0	45	45	-10	-15

Which one would you choose?

Choice - Critiques to rationality

- experiments on two groups of students: in the first group 58% selected '(a)' while in the second group nobody chose '(c)'
- when both choice problems, one after the other, were posed to the same students: 52% chose '(a)' and 7% '(c)'
- rationale: we often transfer complicated problem into simpler ones by "canceling" similar elements

Choice - Critiques to rationality

REASON-BASED CHOICE

- example of cameras: one group with two alternatives (same brand, one lower quality for 170€ and the other higher quality 240€), another group with a third alternative (top quality camera for 470€). Guess what happens to the number of people choosing the 240€ camera. . .
- making choice sometimes requires finding motives to pick one alternative over the others
- in this kind of situations, choice is based on "internal reasons", and it is often difficult to reconcile the actual choices with the rational paradigm

Choice - Critiques to rationality

REASON-BASED CHOICE

- example of cameras: one group with two alternatives (same brand, one lower quality for 170€ and the other higher quality 240€), another group with a third alternative (top quality camera for 470€). Guess what happens to the number of people choosing the 240€ camera... Internal reason: compromising alternative
- making choice sometimes requires finding motives to pick one alternative over the others
- in this kind of situations, choice is based on "internal reasons", and it is often difficult to reconcile the actual choices with the rational paradigm

Summing Up

- preferences on X are binary relations on X satisfying completeness and transitivity. They can be formalized in different but equivalent ways

- a system of preferences can be represented by an utility function which has no meaning other than that of representing a preference relation

- a choice problem is seen as a nonempty subset of X and a choice consists in selecting the element of this subset that is optimal wrt the system of preferences considered

Consumer Theory - Introduction

Consumer theory is characterized axiomatically: few "meaningful" assumptions are set forth; the rest of the theory is then developed via the process of deduction

There are four main building blocks:

- CONSUMPTION SET: set of all the alternatives X;
- FEASIBLE SET: set of all the conceivable and realistically obtainable alternatives $B \subset X$ given the consumers' circumstances (income, prices);
- PREFERENCE RELATION: specifies limits, forms of consistencies and inconsistencies of the consumer's ability to make choice among alternatives;
- BEHAVIORAL ASSUMPTION: principles guiding the consumer in his choice.

Preference relations

- we represent the consumer's preference by a binary relation $\succeq$ defined on the consumption set X.

  $x^1 \succeq x^2$ $\qquad$ x^1 is at least as good as x^2

  where $x^i = (x_1, \ldots, x_n)$ is consumption bundle with $x \in X \subseteq \mathbb{R}^n_+$
- using a binary relation implies that the consumer examines only two alternative consumption bundles at a time and makes a decision between those two
- we impose two other important restrictions

Preference - Axioms

COMPLETENESS: For all x^1 and $x^2 \in X$, either $x^1 \succeq x^2$ or $x^2 \succeq x^1$

The consumer has always the ability to discriminate and the necessary knowledge to evaluate alternatives. Philosophically questionable: extreme situations.

TRANSITIVITY: For all x^1, x^2 and $x^3 \in X$ if $x^1 \succeq x^2$ and $x^2 \succeq x^3$ then $x^1 \succeq x^3$

As discussed, this axiom imposes a very peculiar form of consistencies to consumers' choices

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Preferences - Definitions

PREFERENCE RELATION: a binary relation $\succeq$ defined on X is called a preference relation if it satisfies completeness and transitivity

STRICT PREFERENCE: a binary relation $\succ$ defined on X as

$$x^1 \succ x^2 \quad \text{IFF} \quad x^1 \succeq x^2 \text{ and } x^2 \not\succeq x^1$$

is called strict preference relation induced by $\succeq$

INDIFFERENCE: a binary relation $\sim$ defined on X as

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Are SPR and IR transitive? Are they complete?

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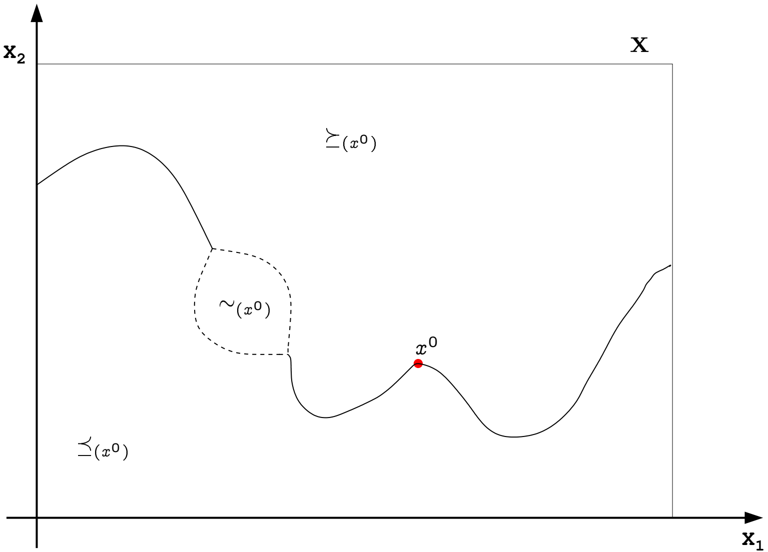
Are SPR and IR transitive? Are they complete?

Preferences - Preference set induced by $\succeq$

Let $x^0 \in X$. Then we can define

- $\succeq_{(x^0)} \equiv \{x | x \in X, x \succeq x^0\}$ "at least as good" set
- $\preceq_{(x^0)} \equiv \{x | x \in X, x \preceq x^0\}$ "no better than" set
- $\succ_{(x^0)} \equiv \{x | x \in X, x \succ x^0\}$ "preferred to" set
- $\prec_{(x^0)} \equiv \{x | x \in X, x \prec x^0\}$ "worse than" set
- $\sim_{(x^0)} \equiv \{x | x \in X, x \sim x^0\}$ "indifference" set

Preferences - Preference set induced by $\succeq$



Preferences - Axioms

CONTINUITY: For all $x \in X$, both $\succeq_{(x)}$ and $\preceq_{(x)}$ are closed sets.

- continuity guarantees that sudden preference reversals do not occur; continuity can be restated as

  if each element y_n of a sequence is $\succeq x$ and $\lim_{n \to \infty} y_n = y$

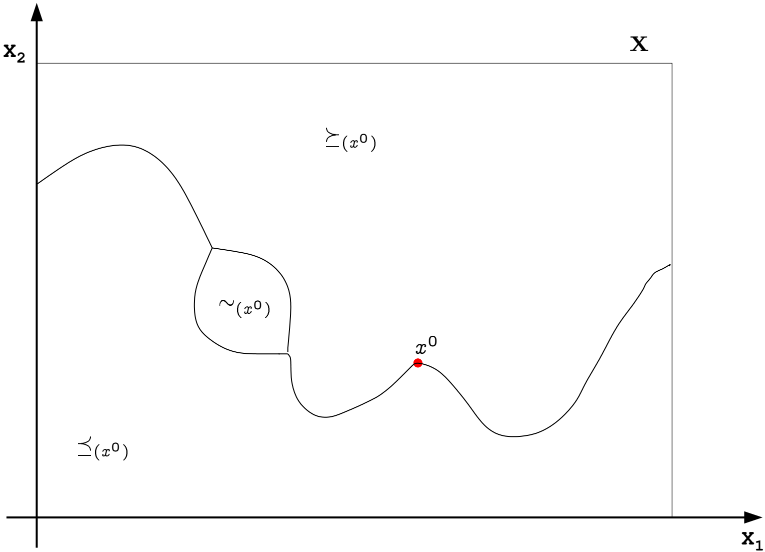
  then

  $$y \succeq x$$

- consequence: if both $\succeq_{(x)}$ and $\preceq_{(x)}$ are closed then also $\sim_{(x)}$ is closed, non open area exists.

Proof.

Let A and B two closed sets. Then A^c and B^c are open.
Unions of open sets are open sets.
De Morgan's Law states that $A^c \cup B^c = (A \cap B)^c$ hence $(A \cap B)^c$ is open and $(A \cap B)$ is closed


X
$\mathbf{x_2}$
$\succeq_{(x^0)}$
$\sim_{(x^0)}$
x^0
$\preceq_{(x^0)}$
$\mathbf{x_1}$

Preferences - Axioms

LOCAL NONSATIATION: For all $x^0 \in X$ and for all $\epsilon > 0$, there exists some $x \in B_\epsilon(x^0) \cap \mathbb{R}^n_+$ such that $x \succ x^0$

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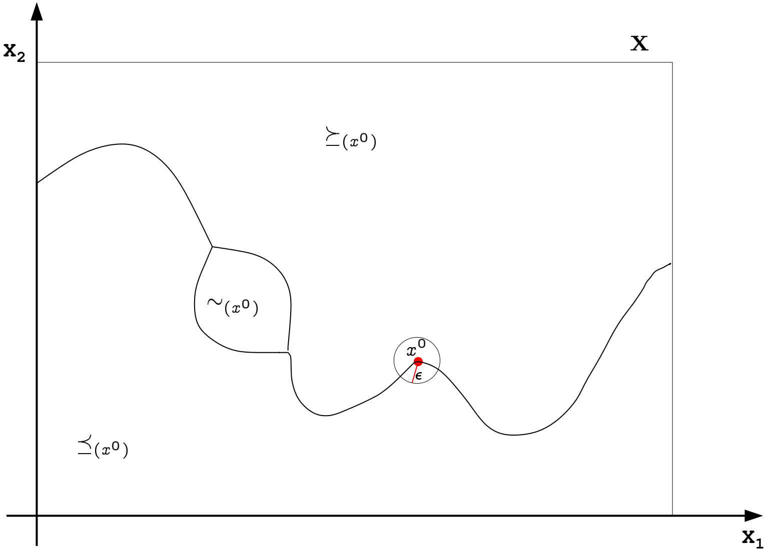

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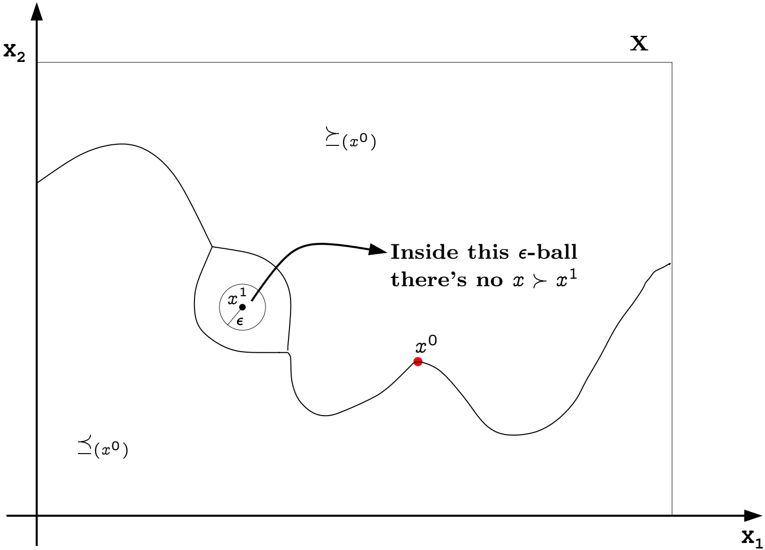

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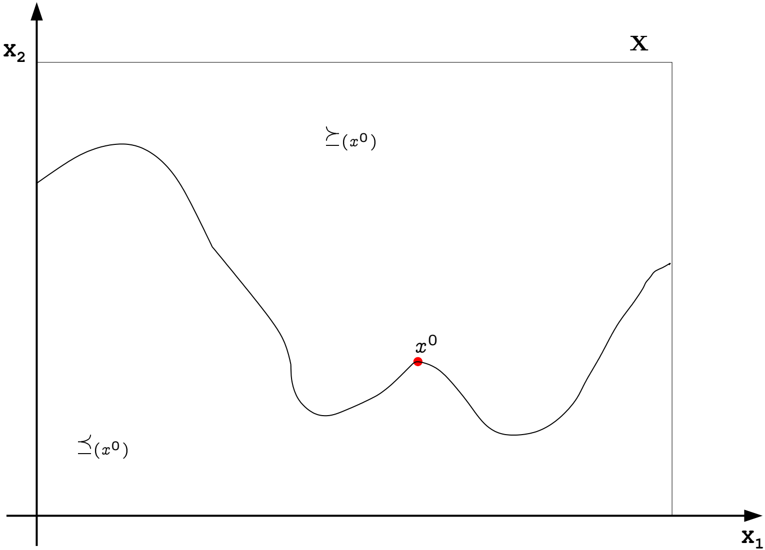

Preferences - Axioms

STRICT MONOTONICITY: For all x^0 and $x^1 \in \mathbb{R}^n_+$ if $x^0 \geq x^1$ then $x^0 \succeq x^1$, while if $x^0 > x^1$ then $x^0 \succ x^1$

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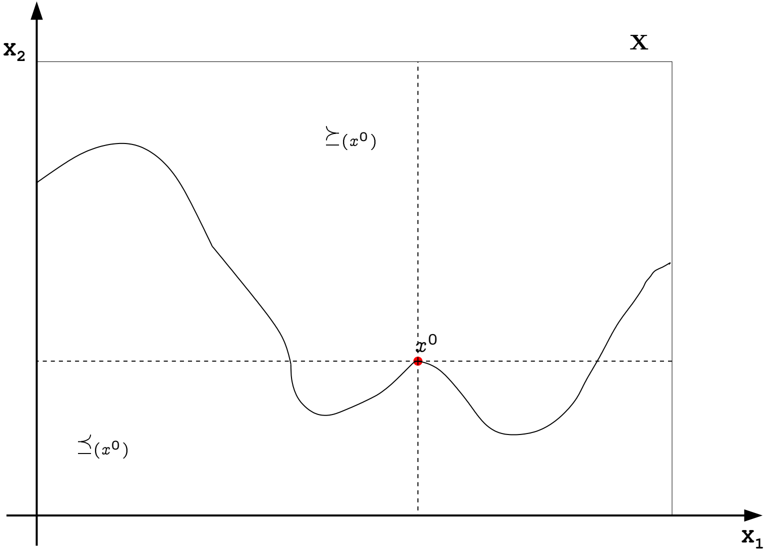

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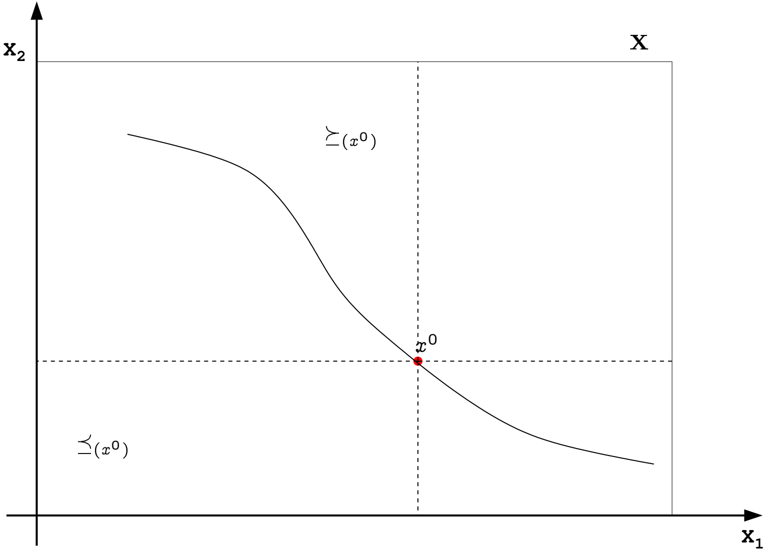

Preferences - Axioms

STRICT CONVEXITY: If $x^0 \neq x^1$ and $x^1 \succeq x^0$ then $tx^1 + (1-t)x^0 \succ x^0 \; \forall t \in (0,1)$

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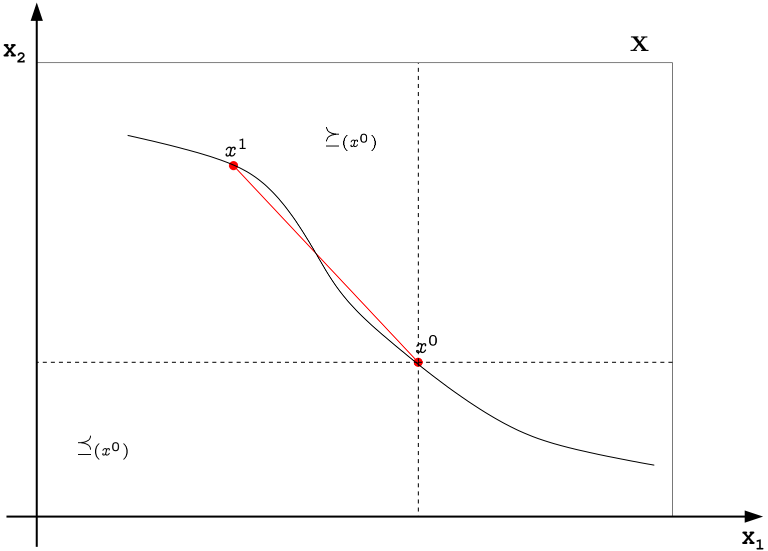

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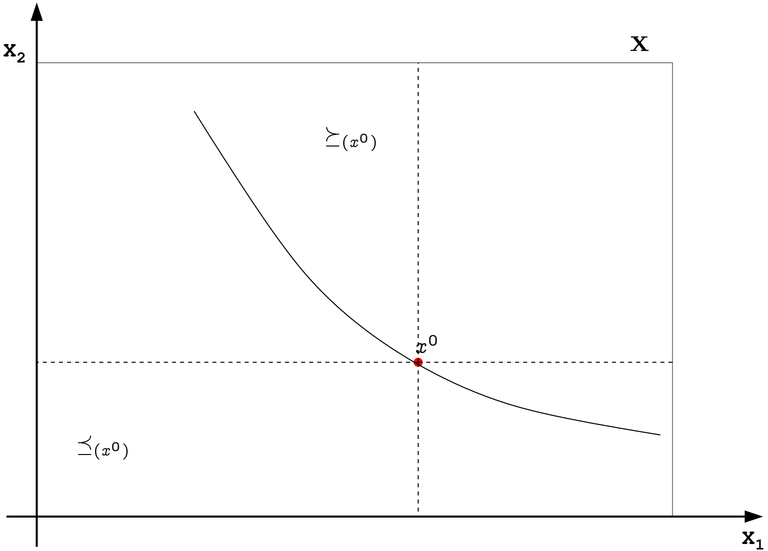

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Intuitively strict convexity can be interpreted in two ways:

- more balanced consumption bundles are preferred
- the rate at which the consumer is willing to give up x_2 in exchange of x_1 (MRS) decreases with x_1

Preferences axioms - Sum up

- Completeness and Transitivity describe a consumer who can make consistent comparisons among alternatives
- Continuity guarantees the existence of topologically nice "$\succeq$" and "$\preceq$" sets
- Strict Monotonicity and Strict Convexity require that tastes display some form of "more is better" and "more balanced is better"

Utility function

In modern economic theory an utility function is a convenient way to summarize the info contained in the consumer's preference relation.

UTILITY FUNCTION: a real valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is called utility function representing the preference relation $\succeq$ if

$$\forall x^0, x^1 \in \mathbb{R}^n_+ \qquad u(x^0) \geq u(x^1) \Leftrightarrow x^0 \succeq x^1$$

Utility function

Theorem

If $\succeq$ is complete, transitive, continuous and strictly monotonic there exists a continuous real valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ which represents $\succeq$.

Utility function

Theorem

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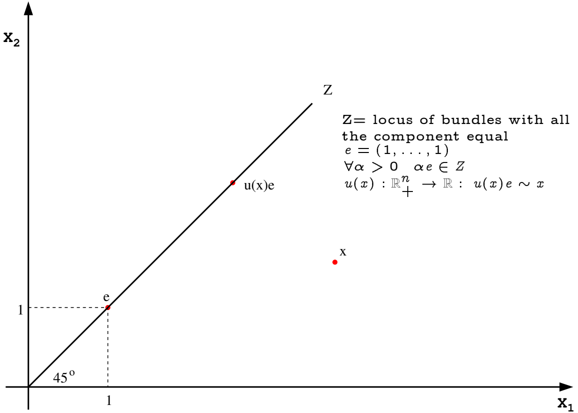


Proof.

1. Does the number $u(x)$ satisfying its definition exist?
2. If yes, is it uniquely determined, so that $u(x)$ is a well-defined function?

Let us define

$$A \equiv \{t \geq 0 | te \succeq x\}$$
$$B \equiv \{t \geq 0 | te \preceq x\}$$

If $t^* \in A \cap B$ then $t^* e \sim x$ and hence $u(x) = t^*$ satisfies our definition. Is $A \cap B$ a non-empty set? Continuity of $\succeq$ implies that both A and B are closed set, then by strict monotonicity

$$t \in A \Rightarrow t' \in A \quad \forall t' \geq t \Rightarrow A = [\bar{t}, +\infty)$$
$$t \in B \Rightarrow t' \in B \quad \forall t' \leq t \Rightarrow B = [0, \underline{t}]$$

Proof. - cont'ed.

Since $\succeq$ is complete it is either $te \succeq x$ or $te \preceq x$ and then $t \in A \cup B$. Finally, since $A \cup B = \mathbb{R}_+$ then $A \cup B = [0, \underline{t}] \cup [\overline{t}, +\infty)$ from which it follows that $\underline{t} \leq \overline{t}$ and hence $A \cap B \neq \emptyset$.

We prove the uniqueness by contraddiction. Suppose there are two numbers satisfying our definition, $u_1(x) \neq u_2(x)$, then

$$u_1(x) = t_1 e \sim x$$
$$u_2(x) = t_2 e \sim x$$

which implies (by transitivity) $t_1 e \sim t_2 e$ and hence $t_1 = t_2$ that is in contraddiction with our assumption.

Then $\forall x \in \mathbb{R}^n_+ \;\; \exists!\, u(x) : u(x)e \sim x$. The proof is concluded by showing that $u(x)$ is continuous and represents $\succeq$.

Proof. - cont'ed.

To this aim, let's consider two consumption bundles x^1 and x^2

$$x^1 \succeq x^2 \Leftrightarrow u(x^1)e \sim x^1 \succeq x^2 \sim u(x^2)e$$
$$\Leftrightarrow u(x^1)e \succeq u(x^2)e \Leftrightarrow u(x^1) \geq u(x^2)$$

To prove that $u(x)$ is continuous we need to show that the inverse image under $u(.)$ of every open ball (a, b) in $\mathbb{R}$ is open in $\mathbb{R}^n_+$. This means

$$\begin{aligned} u^{-1}((a,b)) &= \{x \in \mathbb{R}^n_+ | a < u(x) < b\} \\ &= \{x \in \mathbb{R}^n_+ | ae \prec x \prec be\} \\ &= \succ_{(ae)} \cap \prec_{(be)} \end{aligned}$$

By continuity of $\succeq$ both $\preceq_{(ae)}$ and $\succeq_{(be)}$ are closed, then their complements are open. By De Morgan's Law and since the intersection of open set is an open set it follows that $u(x)$ is continuous.

Utility function

- This theorem is clearly very important since allows us to avoid(!) the primitive set-theoretic preference relation.
- Is the utility function $u(x)$ representing $\succeq$ unique?

Theorem

Let $\succeq$ be a preference relation on $\mathbb{R}^n_+$ and suppose $u(x)$ represent it. Then $v(x) = f(u(x))$ represents $\succeq$ IFF f(.) is strictly increasing on the set of values taken by $u(x)$

Maths refresh

Geometric interpretation of the gradient. Consider a function $u(x) : \mathbb{R}^2 \Rightarrow \mathbb{R}$. Using the chain rule we can compute the rate of change of $u(x_1, x_2)$ at any given point x^* in any given direction $v = (v_1, v_2)$

$$\begin{aligned} x &= x^* + tv \qquad t \in \mathbb{R} \\ g(t) &\equiv u(x^* + tv) = u(x_1^* + tv_1, x_2^* + tv_2) \\ g'(0) &= \left(\tfrac{\partial u}{\partial x_1}(x^*) \quad \tfrac{\partial u}{\partial x_2}(x^*) \right) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = Du(x^*)\, v \end{aligned}$$

Sometimes $Du(x^*)$ is written as a column vector and is called gradient, $\nabla u(x^*)$. Using the dot product

$$Du(x^*)v = \nabla u(x^*) \cdot v = \sum_{i=1}^{2} \frac{\partial u}{\partial x_i}(x^*) v_i = ||\nabla u(x^*)||\; ||v|| \cos\theta$$

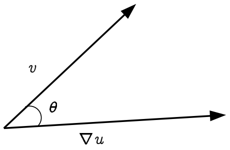
Maths refresh

In what direction does the function $u(x)$ increase most rapidly?

Maths refresh

In what direction does the function $u(x)$ increase most rapidly?

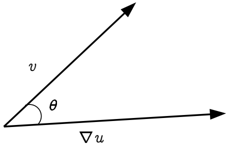
$$\nabla u(x^*) \cdot v = ||\nabla u(x^*)|| \; ||v|| \cos\theta$$



Maths refresh

In what direction does the function $u(x)$ increase most rapidly?

$$\nabla u(x^*) \cdot v = ||\nabla u(x^*)||\,||v||\cos\theta$$



Clearly $\nabla u(x^*) \cdot v$ is maximized when $\cos\theta = 1$, that is when $\theta = 0^o$

Theorem

Let $u(x) : D \to \mathbb{R}$ be a C^1 function. At any $x \in D$ for which $\nabla u(x) \neq 0$ the gradient $\nabla u(x)$ points into the direction in which $u(x)$ increases most rapidly.

Maths refresh

Theorem (Dini's)

Let $u(x_1, x_2)$ be a C^1 function on a ball about $(x_1^0, x_2^0) \in \mathbb{R}^2$. Suppose that $u(x_1^0, x_2^0) = c$ and consider the level curve

$$u(x_1, x_2) = c \quad .$$

If $\partial u / \partial x_2 \neq 0$, there exists a function $x_2 = x_2(x_1)$ defined in an interval I around x_1^0 such that

$$\begin{aligned} u(x_1, x_2(x_1)) &\equiv c \quad \forall x \in I \\ x_2(x_1^0) &= x_2^0 \\ x_2'(x_1^0) &= -\frac{\frac{\partial u}{\partial x_1}(x_1^0, x_2^0)}{\frac{\partial u}{\partial x_2}(x_1^0, x_2^0)} \end{aligned}$$

Maths refresh

Theorem (Gradient and level curves)

Let $u(x_1, x_2)$ *be a* C^1 *function on a ball about* $(x_1^0, x_2^0) \in \mathbb{R}^2$*. Suppose that* (x_1^0, x_2^0) *is a regular point of* u *(either* $\partial u/\partial x_1(x_1^0, x_2^0) \neq 0$ *or* $\partial u/\partial x_2(x_1^0, x_2^0) \neq 0$*). Then*

$$\nabla u(x_1^0, x_2^0) \quad \perp \quad u(x_1^0, x_2^0) = c \quad .$$

Maths refresh

Proof.(sketch).

In general the directional derivative is given by

$$Du(x^*)v = \nabla u(x^*) \cdot v \quad .$$

If we want to move along the level curve it must be

$$\begin{pmatrix} \frac{\partial u}{\partial x_1}(x_1^0, x_2^0) \\ \frac{\partial u}{\partial x_2}(x_1^0, x_2^0) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -\frac{\frac{\partial u}{\partial x_1}(x_1^0, x_2^0)}{\frac{\partial u}{\partial x_2}(x_1^0, x_2^0)} \end{pmatrix} = 0$$

and since

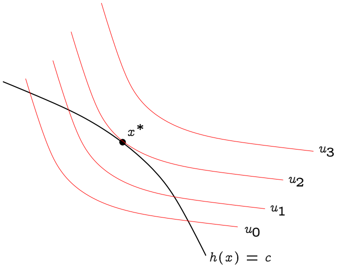
$$\nabla u \cdot v = ||\nabla u(x)|| \; ||v|| \cos\theta \tag{1}$$

we get that θ, the angle between the gradient and the tangent to the level curve must be equal to 90^o.

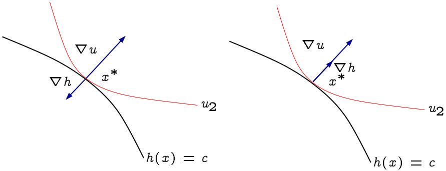
Maths refresh

Geometric intuition behind the Lagrangian. Suppose to have the following problem

$$\underset{x}{\text{MAX}} \quad u(x)$$
$$\text{s.t.} \quad h(x) = c$$



In x^* the level curves and the constraint present the same slope, hence



Maths refresh

In either cases the two gradients are scalar multiples of each other. If we call the multiplier λ^* we get

$$\nabla u(x^*) = \lambda^* \nabla h(x^*)$$

that is

$$\begin{cases} \frac{\partial u}{\partial x_1}(x^*) = \lambda^* \frac{\partial h}{\partial x_1}(x^*) \\ \frac{\partial u}{\partial x_2}(x^*) = \lambda^* \frac{\partial h}{\partial x_2}(x^*) \end{cases}$$

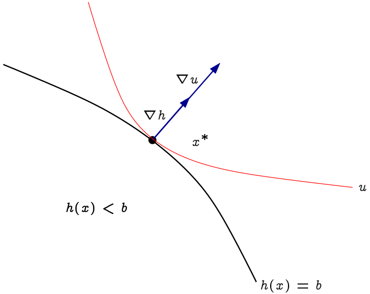
which together with the constraint $h(x) = c$ represent the FOC of our problem. These conditions can be obtained by defining the Lagrangian $L = u(x) - \lambda(h(x) - c)$ and requiring that

$$\begin{cases} \frac{\partial L}{\partial x_1}(x^*) = 0 \\ \frac{\partial L}{\partial x_2}(x^*) = 0 \\ \frac{\partial L}{\partial \lambda}(x^*) = 0 \end{cases}$$

Maths refresh

Inequality constraint. Suppose to have the following problem

$$\underset{x}{\text{MAX}} \quad u(x)$$
$$\text{s.t.} \quad h(x) \leq b$$



If $h(x^*) = b$ again x^* the level curve and the constraint present the same slope. However,

1. since the constraint is $h(x) \leq b$ we know that $\nabla h(x)$ must point to the region where $h(x) > b$
2. since x^* must maximize $u(x)$ we are sure that $\nabla u(x)$ must point to the region where $h(x) > b$

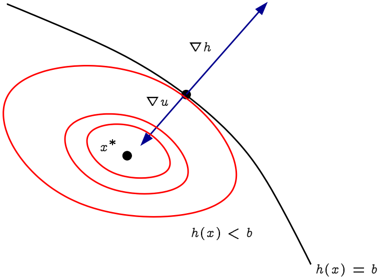
Maths refresh

Hence with the inequality constraint we must add to the usual condition a positivity constraint on λ

$$\begin{cases} \nabla u(x^*) = \lambda^* \nabla h(x^*) \\ \lambda^* \geq 0 \end{cases}$$

Maths refresh

With the inequality constraint we must investigate also what happens if in $h(x^*) < b$. In this case



- $\nabla u(x^*)$ and $\nabla h(x^*)$ point in opposite directions and λ^* in $\nabla u(x^*) = \lambda^* \nabla h(x^*)$ must be negative
- x^* is an unconstrained maxima so the usual FOC $\nabla u(x^*) = 0$ apply

Maths refresh

Lagrangian - Sum up.

$$\underset{x_1,x_2}{\text{MAX}} \quad u(x_1,x_2)$$

$$\text{s.t.} \quad h(x_1,x_2) \leq b$$

$$\begin{cases} \frac{\partial L}{\partial x_1}(x^*) = 0 \\ \frac{\partial L}{\partial x_2}(x^*) = 0 \\ \frac{\partial L}{\partial \lambda}(x^*) = 0 \\ \lambda^*(h(x_1,x_2) - b) = 0 \\ \lambda^* \geq 0 \end{cases}$$

where $L = u(x_1,x_2) - \lambda(h(x_1,x_2) - b)$

Maths refresh

Envelope Theorem. Let $u(x; a)$ and $h(x; a)$ be C^1 $\mathbb{R}^n \text{x} \mathbb{R} \to \mathbb{R}$ functions. Let $x^*(a)$ the solution of

$$\begin{aligned} &\underset{x}{\text{MAX}} \quad u(x) \\ &\text{s.t.} \quad h(x; a) = 0 \quad . \end{aligned}$$

Suppose $x^*(a)$ and $\lambda^*(a)$ are C^1 functions of a, then

$$\frac{d}{da} u(x^*(a); a) = \frac{\partial L}{\partial a}(x^*(a), \lambda^*(a); a)$$

where L is the Lagrangian for this problem.

Maths refresh

Proof.

From the definition of L we get

$$\frac{\partial L}{\partial a}(x^*(a), \lambda^*, a) = \frac{\partial u}{\partial a}(x^*(a); a) - \lambda \frac{\partial h}{\partial a}(x^*(a); a) \quad .$$

Next consider the following

$$\frac{du}{da}(x^*(a), a) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x^*(a); a)\frac{\partial x_i}{\partial a} + \frac{\partial u}{\partial a}(x^*(a); a) \cdot 1$$

$$h(x^*(a); a) = 0 \Rightarrow \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(x^*(a); a)\frac{\partial x_i}{\partial a} + \frac{\partial h}{\partial a}(x^*(a); a) = 0$$

$$\frac{\partial u}{\partial x_i}(x^*(a); a) = \lambda \frac{\partial h}{\partial x_i}(x^*(a); a) \quad \text{FOC of the Max problem}$$

Maths refresh

Proof. - Cont'ed.

Substituting the last two conditions in the definition of du/da we get

$$\frac{du}{da}(x^*(a), a) = \underbrace{\lambda \frac{\partial h}{\partial a}(x^*(a); a) + \frac{\partial u}{\partial a}(x^*(a); a)}_{\frac{\partial L}{\partial a}(x^*(a), \lambda^*(a); a)} \quad .$$

□

At the most abstract level we view the consumer as having

1. consumption set $\mathbb{R}^n_+$
2. preference relation $\succeq$ defined on $\mathbb{R}^n_+$
3. circumstances (income and prices) that limit the alternatives the consumer is able to achieve thus defining a feasible set $B \subset \mathbb{R}^n_+$
4. motivation to obtain the most preferred alternative according to his preferences

Assumption.

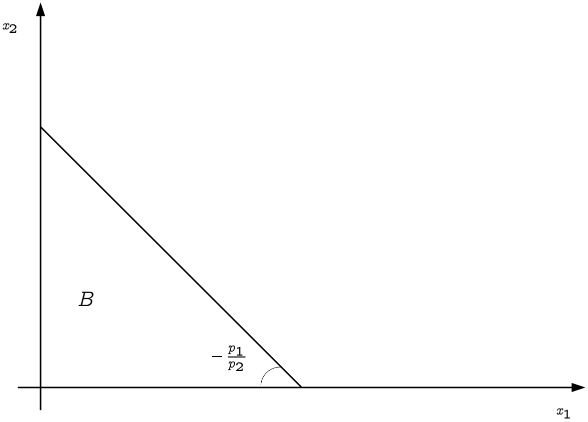
The consumer's preference relation $\succeq$ is complete, transitive, continuous, strictly monotonic and strictly convex in $\mathbb{R}^n_+$. Therefore $\succeq$ can be represented by a real valued utility function, $u(x)$, that is continuous, strictly increasing and strictly quasiconcave on $\mathbb{R}^n_+$.

Budget set

Our consumer operates in a market economy:

1. there are $p_i \quad \forall i$
2. agents cannot influence prices
3. agents are endowed with an income $y > 0$

Budget set: $B = \{x | x \in \mathbb{R}^n_+; p \cdot x \leq y\}$



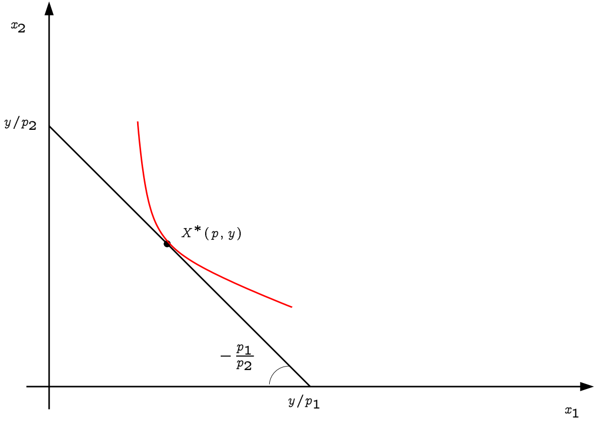
Consumer problem: mathematical structure

$$\underset{x}{\text{MAX}} \quad u(x)$$
$$\text{s.t.} \quad p \cdot x - y \leq 0$$

- $u(x)$ is a continuous real valued function
- B is a non empty, closed, bounded (thus compact) subset of $\mathbb{R}^n$
- $u(x)$ is quasiconcave and B is convex $\Rightarrow$ there exists a unique max (Weierstrass's Theorem)
- preference are strictly monotonic $\Rightarrow$ x^* satisfies the budget constraint with equality

Consumer problem: mathematical structure

$$\underset{x}{\text{MAX}} \quad u(x)$$
$$\text{s.t.} \quad p \cdot x - y \leq 0$$

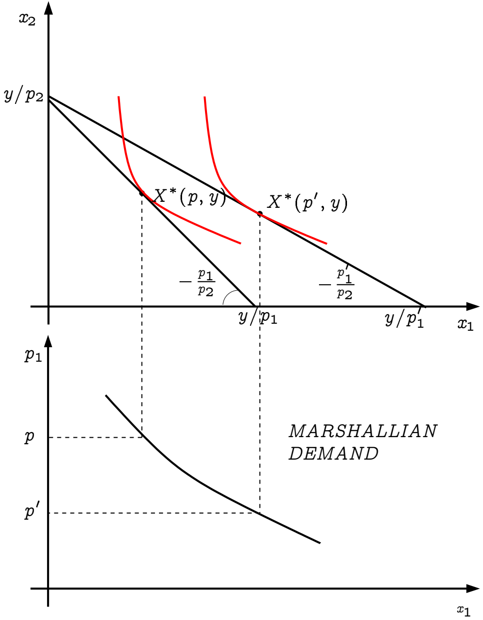


Marshallian demand function

The unique solution to the CP(Consumer Problem) depends on p and y; hence x^* can be viewed as a function of these parameters

$$x^* = x(p, y) \quad \text{Marshallian demand}$$

Marshallian demand function



CP with differentiable utility

$$\underset{x}{\text{MAX}} \quad u(x)$$
$$\text{s.t.} \quad p \cdot x - y \leq 0$$

If $u(x)$ is differentiable we can define the Lagrangian

$$L = u(x) - \lambda(px - y)$$

and set the usual FOC

$$\begin{cases} \frac{\partial L}{\partial x_i}(x^*) = \frac{\partial u}{\partial x_i} - \lambda^* p_i = 0 \quad \forall i \\ \frac{\partial L}{\partial \lambda}(x^*) = px - y = 0 \\ \lambda^*(px^* - y) = 0 \quad \text{[redundant]} \\ \lambda^* \geq 0 \end{cases}$$

CP with differentiable utility

- since $\nabla u(x) > 0$ then from the FOC

$$\frac{\partial u}{\partial x_j}(x^*) = \lambda^* p_j \quad \forall j$$

  at the equilibrium the marginal utility is proportional to price for all goods

- alternatively,

$$MRS_{jk} = -\frac{\partial u/\partial x_j}{\partial u/\partial x_k} = -\frac{p_j}{p_k} \quad \forall j, k$$

  the slope of the indifference curve through x^* is equal to the slope of the budget constraint, where MRS_{jk} is the rate at which x_j can be substituted for x_k with no change in consumer's utility

CP - Sufficiency condition

In general the FOC are only necessary conditions. In this particular case however

Theorem

If $u(x)$ is continuous and (quasi)concave on $\mathbb{R}^n_+$ and $(p, y) > 0$ then if $u(x)$ is differentiable at x^* and $(x^*, \lambda^*) > 0$ solves

$$\frac{\partial u(x^*)}{\partial x_i} - \lambda^* p_i = 0 \quad \forall i$$

then x^* solves the CP.

To prove this important theorem we need a preliminary result.

Lemma

Let $u(x)$ be a C^1 function in $\mathbb{R}$. Then $u(x)$ is concave in $I \subset \mathbb{R}$ IFF

$$u(x^1) - u(x^0) \leq u'(x^0)(x^1 - x^0) \quad \forall x^1, x^0 \in I$$

This condition generalizes in

$$\nabla u(x^0)(x^1 - x^0) \geq 0 \quad \textit{if} \quad u(x^1) \geq u(x^0).$$

Geometric intuition.

Proof.

ONLY IF (sufficient condition). If $u(x)$ is concave then

$$
\begin{aligned}
tu(x^1) + (1-t)u(x^0) &\leq u(tx^1 + (1-t)x^0) \\
u(x^1) - u(x^0) &\leq \frac{u(tx^1 + x^0 - tx^0) - u(x^0)}{t(x^1 - x^0)}(x^1 - x^0) \\
u(x^1) - u(x^0) &\leq \lim_{t \to 0} \frac{u(x^0 + t(x^1 - x^0)) - u(x^0)}{t(x^1 - x^0)}(x^1 - x^0) \\
u(x^1) - u(x^0) &\leq u'(x^0)(x^1 - x^0)
\end{aligned}
$$

Proof.

IF (necessary condition). If

$$u(x) - u(y) \leq u'(y)(x-y) \quad \forall x, y \in I \qquad \text{then}$$

$$u(x^1) - u((1-t)x^1 + tx^0) \leq tu'((1-t)x^1 + tx^0)(x^1 - x^0) \qquad [*]$$

$$u(x^0) - u((1-t)x^1 + tx^0) \leq -(1-t)u'((1-t)x^1 + tx^0)(x^1 - x^0) \qquad [**]$$

Multiplying [*] and [**] respectively by $1-t$ and t

$$(1-t)(u(x^1) - u((1-t)x^1 + tx^0)) \leq (1-t)t\ u'((1-t)x^1 + tx^0)(x^1 - x^0)$$

$$t(u(x^0) - u((1-t)x^1 + tx^0)) \leq -t(1-t)\ u'((1-t)x^1 + tx^0)(x^1 - x^0)$$

summing up the two equations

$$(1-t)(u(x^1)) + t(u(x^0)) - u((1-t)x^1 + tx^0) \leq 0$$

$$(1-t)(u(x^1)) + t(u(x^0)) \leq u((1-t)x^1 + tx^0)$$

Proof of the sufficiency theorem.

Let's prove the statement by contradiction. Assume x^* such that $\nabla u(x^*) = \lambda^* p$ and with $px = y$ exists. If it is not the solution of the CP then, $\forall t \in [0,1]$ it must exist an x^0 such that

$$\begin{cases} u(x^0) > u(x^*) \\ px^0 - y \leq 0 \end{cases} \quad \overset{\Longrightarrow}{\text{continuity of } u(x)} \quad \begin{cases} u(tx^0) > u(x^*) \\ p(tx^0) - y < 0 \end{cases}$$

Setting $x^1 = tx^0$ one gets

$$\begin{aligned} \nabla u(x^*)(x^1 - x^*) &= \lambda^* p(x^1 - x^*) \\ &= \lambda^*(px^1 - px^*) \\ &< \lambda^*(y - y) = 0 \end{aligned}$$

which, by the previous Lemma, is in contradiction with the assumption of concavity of $u(x)$.

□

Indirect utility function

The value of $u(x^*)$ reached in the solution of the CP depends on the level of prices p and on the consumer's income y. The relations among p, y and $u(x^*)$ can be summarized by

$$v : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R} \qquad v(p, y) = \underset{x \in \mathbb{R}^n_+}{\text{Max}}\, u(x) \quad \text{s.t. } px - y \leq 0 \quad .$$

$v(p, y)$ is called indirect utility function and under usual regularity conditions

$$v(p, y) = u(x^m(p, y)) \quad ,$$

where $x^m(p, y)$ is a well defined Marshallian demand function.

Theorem (Properties of $v(p, y)$)

If $u(x)$ is continuous, strictly increasing and differentiable and $(p, y) > 0$ the the associated $v(p, y)$ is

1. *continuous on $\mathbb{R}^n_+ \times \mathbb{R}_+$*
2. *homogeneous of degree 0 in (p, y)*
3. *strictly increasing in y*
4. *decreasing in p*
5. *Roy's Identity:*

$$x_i(p^0, y^0) = -\frac{\partial v(p^0, y^0)/\partial p_i}{\partial v(p^0, y^0)/\partial y}$$

Proof (sketch).

1. Theorem of the maximum states (approximately) that if $u(x)$ and the constraints in the CP are continuous in the parameters and their domain is compact then $v(p, y)$ is continuous

2. Homogeneity of degree 0 means $v(p, y) = v(tp, ty) \quad \forall t > 0$. Easy: does the budget constraint vary?

3. From the Envelope theorem

$$\frac{dv(y,p)}{dy} = \frac{\partial L(p,y,\lambda)}{\partial y} = \lambda > 0$$

4. Similarly

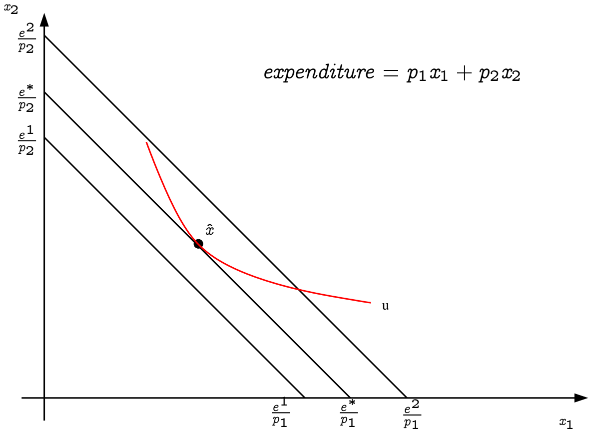
$$\frac{dv(y,p)}{dp_i} = \frac{\partial L(p,y,\lambda)}{\partial p_i} = -\lambda x_i < 0$$

5. From 3. and 4. we get directly Roy's Identity.

Expenditure function

The expenditure function originates fixing the prices of goods and asking at those prices what's the minimum level of money expenditure the consumer must make to achieve a given level of utility.

Expenditure function



Expenditure function

The expenditure function $e(p, u)$ is defined as

$$e : \mathbb{R}^n_+ \times \mathcal{U} \to \mathbb{R}_+ \qquad e(p, u) \equiv \underset{x \in \mathbb{R}^n_+}{\text{Min}}\, px \quad \text{s.t. } u(x) \geq u \quad .$$

Remarks on $e(p, u)$

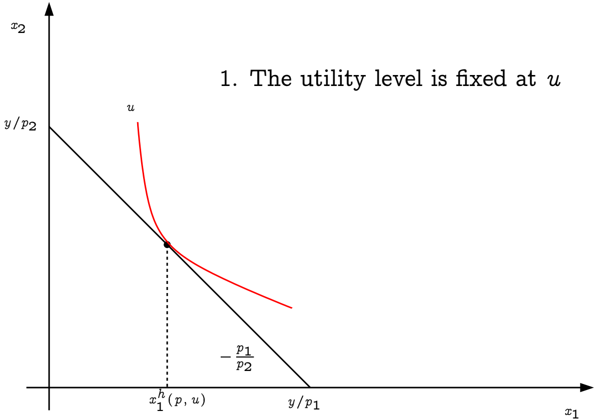
- since $px \geq 0$ and $p > 0$ then the set $\{e | e = p\ x$ for some x with $u(x) \geq u\}$ is closed and $e(p, u)$ is the smallest element of the set
- if $u(x)$ is quasiconcave then the "min" is unique and we can denote the solution to the minimization with a function $\hat{x}(p, u) \geq 0$ and as before

$$e(p, u) = p\ \hat{x}(p, u)$$

- how can we interpret $\hat{x}(p, u)$?

Hicksian demand function

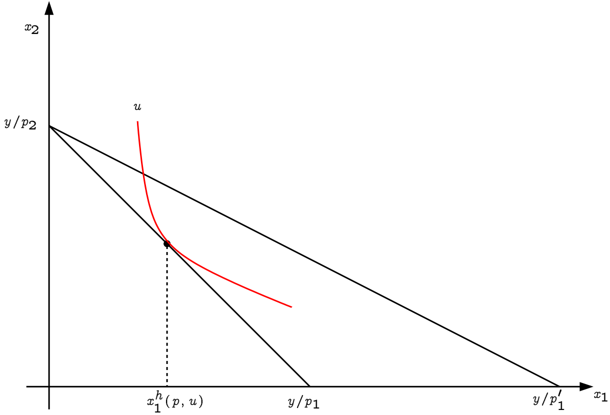
Consider a standard CP, but fix the utility at level u.



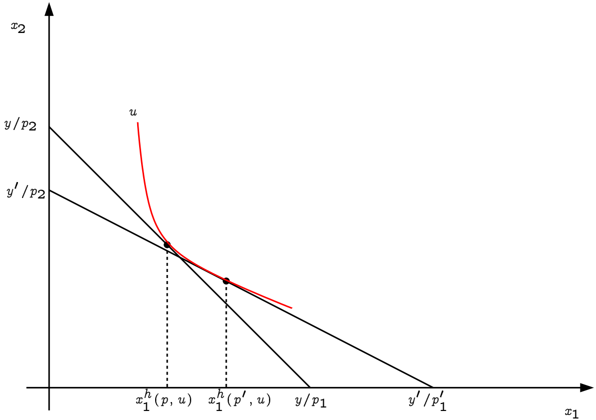
Hicksian demand function

Next, imagine to lower p_1 and to penalize the consumer reducing his income exactly of the amount needed to bring the consumer back to the previous utility level

Hicksian demand function



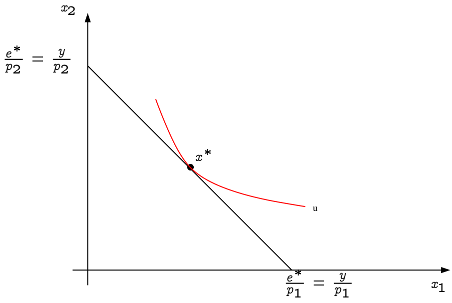
Hicksian demand function



$x^h(p', u)$ represents the consumption bundle chosen by the consumer at the new price level p' after being compensated to maintain constant his level of utility at u. $x^h(p, u)$ are called Hicksian(compensated) demand functions.

Expenditure-minimizing choice and Hicksian demand

The solution $\hat{x}(p, u)$ of the expenditure-minimization problem is the vector of the Hicksian demands $x^h(p, u)$.



EXPLANATION. Each of the hypothetical(compensated) budget constraint implies a level of expenditure exactly equal to minimum level necessary, at given prices, to achieve the utility level u

Theorem (Properties of $e(p, u)$)

If $u(x)$ is continuous and strictly increasing then $e(p, u)$ is

1. 0 *when $u(x)$ takes on the lowest level of utility in $\mathcal{U}$*
2. *continuous on $\mathbb{R}_+ \times \mathcal{U}$*
3. *increasing and unbounded in u* $\forall p > 0$
4. *increasing in p*
5. *homogeneous of degree* 1 *in p*
6. *concave in p*

and if $u(x)$ is strictly quasi-concave

7. *Shephard's Lemma:*

$$\frac{\partial e(p^0, u^0)}{\partial p_i} = x_i^h(p^0, y^0)$$

Proof (sketch).

1. Since $u(x)$ is strictly increasing in x, its lowest value is $u(0)$. Then $px = 0$ and consequently $e(p, u(0)) = 0$.

2. It follows from the theorem of the maximum

3. Let's consider the Lagrangian $L = px - \lambda(-u(x) + u)$. If we assume that the constraint is binding and apply the Envelope theorem

$$\frac{\partial L}{\partial x_i} = 0 \quad \Rightarrow \quad p_i = -\lambda \frac{\partial u(x)}{\partial x_i} \quad \Rightarrow \quad \lambda < 0$$

and hence by the Envelop th.

$$\frac{d\ e(p,u)}{du} = \frac{\partial L}{\partial u} = -\lambda > 0$$

4. cfr. property 7.

Proof.(cont'ed).

5. It suffices to show that if x^* minimizes the expenditure at price p it also minimizes the expenditure at price tp. Suppose not, there exists x' such that

$$tpx' < tpx^* \quad \Rightarrow \quad px' < px \quad \text{contraddiction!}$$

6. Concavity in p requires

$$te(p^0, u) + (1-t)e(p^1, u) \leq e(tp^0 + (1-t)p^1, u)$$

Now consider the following three expenditure minimizing combinations (p^1, x^1), (p^2, x^2) and (p^3, x^3) where $p^3 = tp^1 + (1-t)p^2$. Since x^3 is not necessarily minimizing the expenditure with p^1 and p^2 then

$$\begin{aligned} e(p^3, u) = p^3\, x^3 &= tp^1 1x^3 + (1-t)p^2 x^3 \\ &\geq te(p^1, u) + (1-t)e(p^2, u) \end{aligned}$$

Proof.(cont'ed).

7. By the Envelope theorem

$$\frac{d\ e(p,u)}{dp_i} = \frac{\partial L}{\partial p_i} = x_i^* \equiv x_i^h$$

□

The relation between $v(p, y)$ and $e(p, u)$

Fixing p and y we get

$$u = v(p, y) \quad \text{then by definition} \quad e(p, v(p, y)) \leq y$$

and analogously fixing p and u

$$y = e(p, u) \quad \text{then by definition} \quad v(p, e(p, u)) \geq u \quad .$$

Theorem

Let $v(p, y)$ and $e(p, u)$ be the indirect utility and the expenditure function for a consumer whose utility $u(x)$ is continuous and strictly increasing. Then $\forall p > 0$, $\forall y \geq 0$ and $\forall u \in \mathcal{U}$

$$i)\ e(p, v(p, y)) = y \qquad ii)\ v(p, e(p, u)) = u$$

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$$i)\ e(p, v(p, y)) = y \qquad\qquad ii)\ v(p, e(p, u)) = u$$

Proof.

i) Consider that $\mathcal{U} = [u(0), \bar{x}]$ with $\bar{x}$ either finite or $+\infty$.
Suppose that i) does not hold

$$e(p, v(p, y)) < y \quad \Rightarrow \quad e(p, u) < y \;\text{ with } u = v(p, y)$$

[by continuity of] $u(x)$ $\quad e(p, u + \epsilon) < y \quad \epsilon > 0$

[hence there exists] $y_\epsilon : y_\epsilon = e(p, u + \epsilon) < y$

[since $v(p, y)$ is increasing in y] $v(p, y) > v(p, y_\epsilon) \geq u + \epsilon$

$\Rightarrow u \geq u + \epsilon$ which contradicts $\epsilon > 0$.

ii) Again by contraddiction. Assume that

$$v(p, e(p, u)) > u \quad \text{with } e(p, u) = y$$

[since e() is increasing in u] $y = e(p, u) > e(p, u(0)) = 0 \;\Rightarrow\; y > 0$

[by continuity of $v()$] $\exists \epsilon > 0 : v(p, y - \epsilon) > u \;\Rightarrow\; e(p, u) \leq y - \epsilon$

[which together with] $e(p, u) = y$ generate the contradiction

Duality of demand function

Theorem

Under the usual assumptions, $\forall p > 0$, $\forall y \geq 0$ *and* $\forall u \in \mathcal{U}$ *we have*

$$\begin{aligned} i)\ x_i(p, y) &= x_i^h(p, v(p, y)) \\ ii)\ x_i^h(p, u) &= x_i(p, e(p, u)) \quad . \end{aligned}$$

This theorem states that the solution to the "Max u" problem is also a solution of the "Min px" problem and viceversa.

Proof.

Usual assumption guarantees the existence and uniqueness of the solution for the two problem $\Rightarrow v(p,y)$ and $e(p,u)$ are well defined. Assumes x^0 solves Max $u(x)$ $s.t.$ $px - y \leq 0$, then we have

$$x^0 = x(p, y^0) \quad u(x^0) = u^0$$
$$v(p, y^0) = u^0 \quad px^0 = y^0 \qquad \text{[by definition of } v()\text{]}$$
$$e(p, u^0) = e(p, v(p, y^0)) = y^0 \ , \qquad \text{[by the previous theorem]}$$

which states that x^0 is a solution of the Min px st $u(x) \geq u^0$ and hence

$$x^0 = x^h(p, u^0) = x^h(p, v(p, y^0))$$

Analogously for ii).

□

Income and prices

People do (should) care about real commodities.

- RELATIVE PRICE: number of units of some other commodity that must be foregone to acquire 1 unit of a given good

$$\frac{p_i}{p_j} = \frac{€/unit\ i}{€/unit\ j} = \frac{unit\ j}{unit\ i} \quad \text{relative price of good i}$$

- REAL INCOME: maximum number of units of some commodity the consumer could acquire if he spends his entire money income

$$\frac{y}{p_i} = \frac{€}{€/unit\ i} = unit\ i$$

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Homogeneity and budget balancedness(BB)

Theorem

Under usual conditions the demand function $x_i(p, y)$ $\forall i$ is homogeneous of degree zero in all prices and income and satisfies the budget balancedness

$$px(p, y) = y \qquad \forall (p, y)$$

Proof.

From the properties of the indirect utility we know

$$v(p, y) = v(tp, ty) \Leftrightarrow u(x(p, y)) = u(x(tp, ty))$$

and since the budget sets with (p, y) and (tp, ty) are equal then both $x(p, y)$ and $x(tp, ty)$ were feasible when the other was chosen.

Proof. (cont'ed).

Finally the previous equality and from strict quasi-concavity of $u(x)$ we get

$$x(p, y) = x(tp, ty)$$

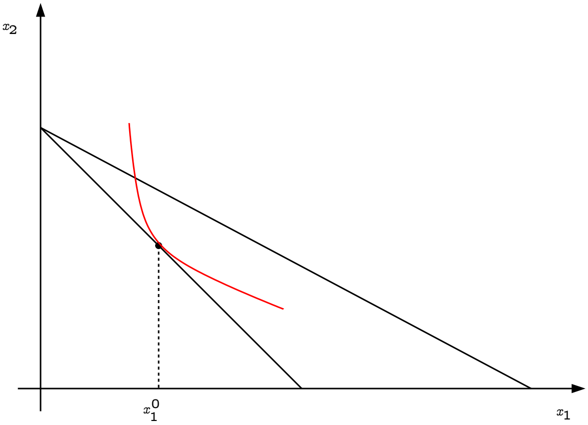
The second part follows directly from the fact that $u(x)$ is strictly increasing.

$\square$

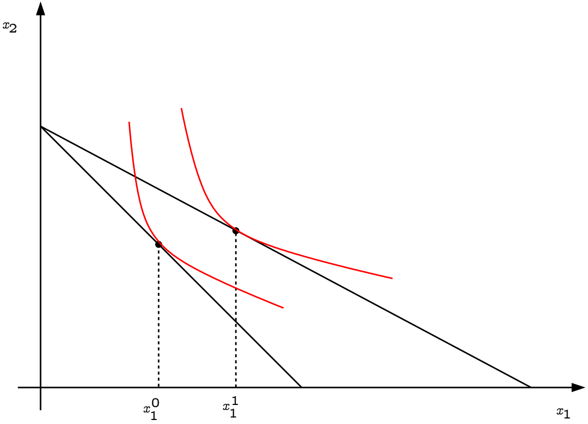
Effects of a price change

What response should we expect in the demanded quantity of a commodity when its price changes? Suppose we observe a reduction in the price of x_1

Effects of a price change



Effects of a price change



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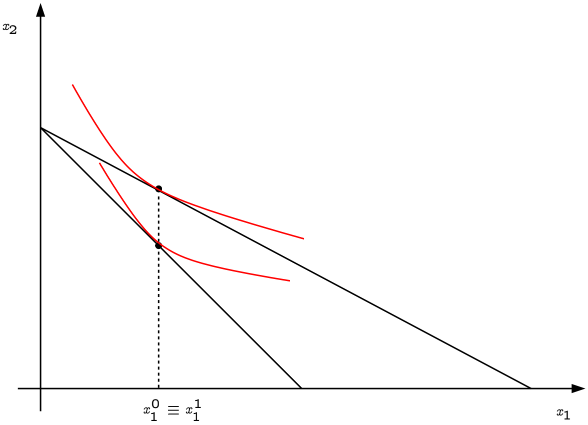

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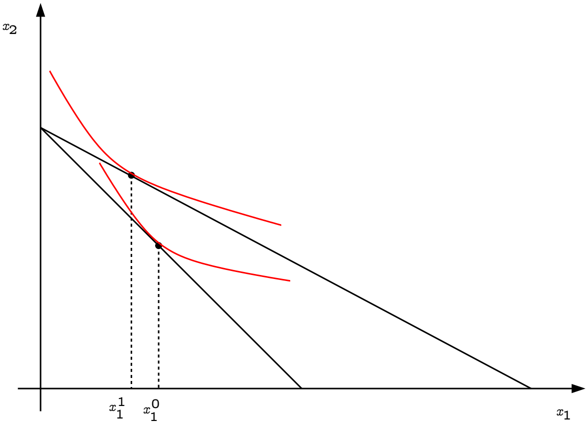

Effects of a price change

Intuitively when a price changes there are two reasons why we should expects a change in the corresponding demand:

- the good whose price declined becomes relatively cheaper compared to the other good; since they are all desiderable we expect the consumer to substitute cheaper good with the more expensive one [SUBSTITUTION EFFECT]
- when the price of any good declines the consumer total command over all goods is increased allowing him to increase his consumption of all good in the way he sees fit [INCOME EFFECT]

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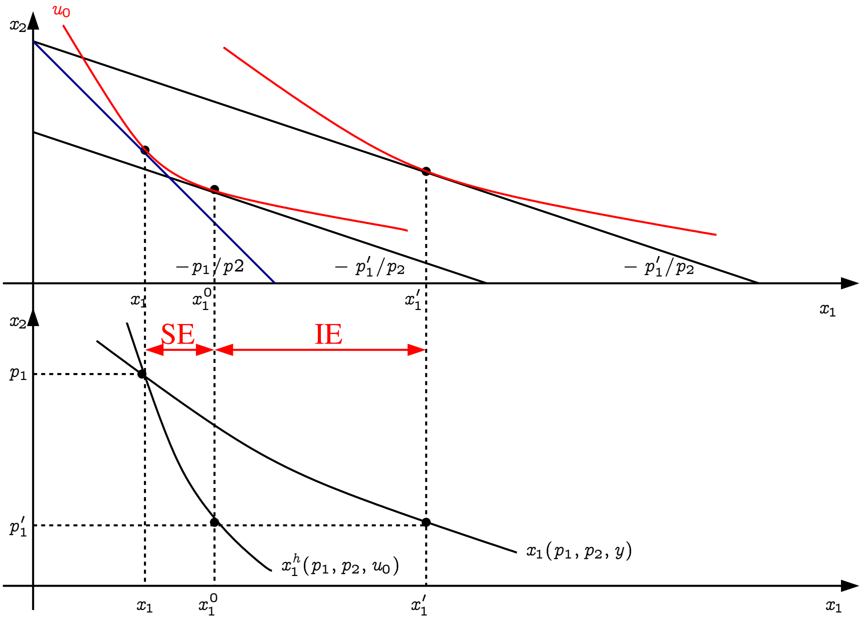
Suppose for example to observe a reduction of the price of x_1 from p_1 to p_1'



u_0
x_2
$-p_1/p2$
$-p_1'/p_2$
$-p_1'/p_2$
x_1
x_1^0
x_1'
x_1
x_2
SE
IE
p_1
p_1'
$x_1(p_1, p_2, y)$
$x_1^h(p_1, p_2, u_0)$
x_1
x_1^0
x_1'
x_1

Theorem (Slustky equation)

Let $x(p, y)$ be the marshallian demand system of a consumer. Let u^* be the utility level the consumer achieves at price p and income y. Then

$$\frac{\partial x_i(p, y)}{\partial p_j} = \frac{\partial x_i^h(p, u^*)}{\partial p_j} - x_j(p, y)\frac{\partial x_i(p, y)}{\partial y} \quad \forall i, j$$

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Proof.

Consider

$$x_i^h(p,u^*) = x_i(p, e(p,u^*))$$

$$\frac{\partial x_i^h(p,u^*)}{\partial p_j} = \frac{\partial x_i(p,e(p,u^*))}{\partial p_j} + \frac{\partial x_i(p,e(p,u^*))}{\partial y}\frac{\partial e(p,u^*)}{\partial p_j}$$

[by total differentiation]

Proof. (cont'ed).

Now by the Shephard's Lemma and by considering that $e(p, u^*) = e(p, v(p, y)) = y$ we get

$$\frac{\partial x_i(p, y)}{\partial p_j} = \frac{\partial x_i^h(p, u^*)}{\partial p_j} - x_j(p, y)\frac{\partial x_i(p, y)}{\partial y}$$

□

REMARK. Note that the Hicksian demands are unobservable. What can we know about Hicksian demands if we cannot even directly see them? Quite a bit.

Proof. (cont'ed).

Now by the Shephard's Lemma and by considering that $e(p, u^*) = e(p, v(p, y)) = y$ we get

$$\frac{\partial x_i(p, y)}{\partial p_j} = \frac{\partial x_i^h(p, u^*)}{\partial p_j} - x_j(p, y)\frac{\partial x_i(p, y)}{\partial y}$$

□

REMARK. Note that the Hicksian demands are unobservable. What can we know about Hicksian demands if we cannot even directly see them? Quite a bit.

Theorem (Negative own-substitution terms)

Let $x_i^h(p,u)$ *be the hisckian demand for good* i*. Then*

$$\frac{\partial x_i^h(p,u)}{\partial p_i} \leq 0 \quad \forall i$$

Proof.

From the Shephard's lemma we have

$$\frac{\partial e(p,u)}{\partial p_i} = x_i^h(p,u)$$

$$\frac{\partial^2 e(p,u)}{\partial p_i^2} = \frac{\partial x_i^h(p,u)}{\partial p_i} \leq 0 \qquad \text{[since } e(p,u) \text{ is concave in } p\text{]}$$

$\square$

IMPLICATION: Law of Demand for normal good.

Theorem (Symmetric substitution term)

Let $x_i^h(p, u)$ be the hiscksian demand for good i. Then

$$\frac{\partial x_i^h(p,u)}{\partial p_j} = \frac{\partial x_j^h(p,u)}{\partial p_i} \quad \forall i, j$$

Proof.

From the Shephard's lemma we have

$$\frac{\partial e(p,u)}{\partial p_i} = x_i^h(p,u) \Rightarrow \frac{\partial x_i^h(p,u)}{\partial p_j} = \frac{\partial^2 e(p,u)}{\partial p_j p_i}$$

$$\frac{\partial^2 e(p,u)}{\partial p_j p_i} = \frac{\partial^2 e(p,u)}{\partial p_j p_i} \qquad \text{[by Young's theorem]}$$

$$\frac{\partial x_i^h(p,u)}{\partial p_j} = \frac{\partial x_j^h(p,u)}{\partial p_i} \quad \forall i, j$$

□

Theorem (Negative semi-definite substitution matrix)

Let $x_i^h(p,u)$ be the hisscksian demand for good i and let

$$\sigma(p,u) \equiv \begin{bmatrix} \frac{\partial x_1^h(p,u)}{\partial p_1} & \cdots & \frac{\partial x_1^h(p,u)}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n^h(p,u)}{\partial p_1} & \cdots & \frac{\partial x_n^h(p,u)}{\partial p_n} \end{bmatrix}$$

be the sibstitution matrix. Then $\sigma(p,u)$ is negative semidefinite.

Proof.

For each element of $\sigma(p,u)$ it holds that

$$\frac{\partial x_i^h(p,u)}{\partial p_j} = \frac{\partial^2 e(p,u)}{\partial p_j p_i} \quad .$$

The theorem then follows from the concavity of $e(p,u)$.

Theorem (Slutsky matrix)

Let $x(p, y)$ be the marshallian demand system. Define the Slustky matrix as

$$\begin{bmatrix} \frac{\partial x_1(p,y)}{\partial p_1} + x_1(p, y)\frac{\partial x_1(p,y)}{\partial y} & \dots & \frac{\partial x_1(p,y)}{\partial p_n} + x_1(p, y)\frac{\partial x_1(p,y)}{\partial y} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n(p,y)}{\partial p_1} + x_n(p, y)\frac{\partial x_n(p,y)}{\partial y} & \dots & \frac{\partial x_n(p,y)}{\partial p_n} + x_n(p, y)\frac{\partial x_n(p,y)}{\partial y} \end{bmatrix}.$$

Then the Slustky matrix $s(p, y)$ is symmetric and negative semi-definite.

Proof.

Consider the Slustky equation

$$\frac{\partial x_i^h(p, u^*)}{\partial p_j} = \frac{\partial x_i(p, y)}{\partial p_j} + x_j(p, y)\frac{\partial x_i(p, y)}{\partial y}$$

then $\sigma(p, u) = s(p, y)$ and hence $s(p, y)$ is symmetric and negative semi-definite.

Testing the theory I

If a marshallian demand system is to be viewed as to a price-taking utility-maximizing consumer then

1. demands must be homogeneous
2. demands must satisfy BB
3. the associated Slutsky matrix must be symmetric and negative semi-definite

These requirements provide a set of restrictions on allowable values for the parameters in any empirically estimated marshallian demand system.

Elasticities

We define

$$\eta_i \equiv \frac{\partial x_i(p,y)}{\partial y}\frac{y}{x_i(p,y)} \quad \text{[income elasticity]}$$

$$\epsilon_{ij} \equiv \frac{\partial x_i(p,y)}{\partial p_j}\frac{p_j}{x_i(p,y)} \quad \text{[cross-price elasticity]}$$

$$s_i \equiv \frac{p_i x_i(p,y)}{y} \quad s_i \geq 0 \quad \sum_{i=1}^{n} s_i = 1$$

Theorem

Aggregations The following aggregating relations hold

$$\text{i)} \sum_{i=1}^{n} \eta_i s_i = 1 \quad \text{[Engel Aggregation]}$$

$$\text{ii)} \sum_{i=1}^{n} \epsilon_{ij} s_i = -s_j \quad \text{[Cournot Aggregation]}$$

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Proof.

i) Differentiating the BB wrt y

$$1 = \sum_{i=1}^{n} p_i \frac{\partial x_i(p,y)}{\partial y} = \sum_{i=1}^{n} p_i \frac{x_i}{y} \frac{\partial x_i(p,y)}{\partial y} \frac{y}{x_i} = \sum_{i=1}^{n} \eta_i s_i \quad .$$

ii) Differentiating the BB wrt p_j

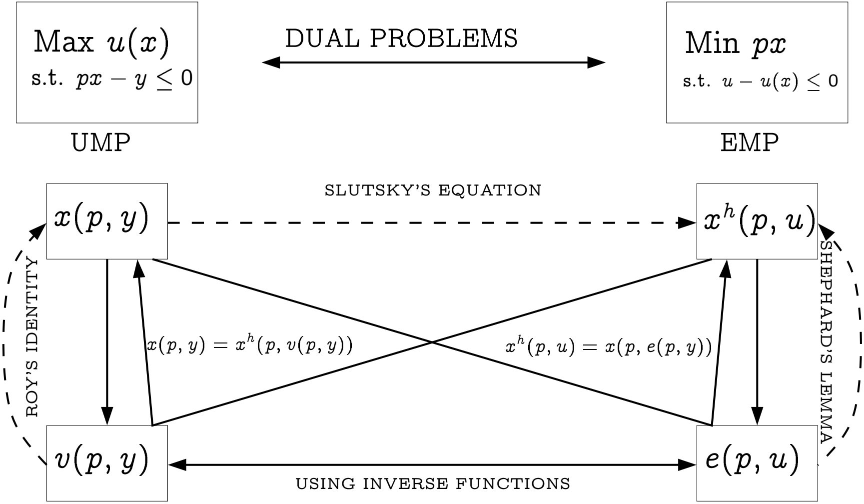
$$0 = \sum_{i \neq j} p_i \frac{\partial x_i(p,y)}{\partial p_j} + x_j(p,y) + p_j \frac{\partial x_j(p,y)}{\partial p_j}$$

$$0 = \sum_{i=1}^{n} p_i \frac{\partial x_i(p,y)}{\partial p_j} + x_j(p,y)$$

$$\sum_{i=1}^{n} p_i \frac{x_i}{y} \frac{\partial x_i(p,y)}{\partial p_j} \frac{p_j}{x_i} = -x_j(p,y) \frac{p_j}{y}$$

$$\sum_{i=1}^{n} \epsilon_{ij} s_i = -s_j$$

□



RI: $x_i(p, y) = -\dfrac{\partial v(p, y)/\partial p_i}{\partial v(p, y)/\partial y}$ SL: $\dfrac{\partial e(p, u)}{\partial p_i} = x_i^h(p, u)$

SE: $\dfrac{\partial x_i(p, y)}{\partial p_j} = \dfrac{\partial x_i^h(p, u)}{\partial p_j} - x_j(p, y)\dfrac{\partial x_i(p, y)}{\partial y}$

Intro

The Theory of Demand has been derived by assuming consumers have preferences satisfying some axioms and deducing both demand functions and their properties.

Samuelson(1947): Why not start and finish with observable behaviors?

INTUITION. If the consumer demands one consumption bundle instead of another affordable one then we may say that the first bundle is revealed preferred to the second.

Weak Axiom of Revealed Preferences - WARP

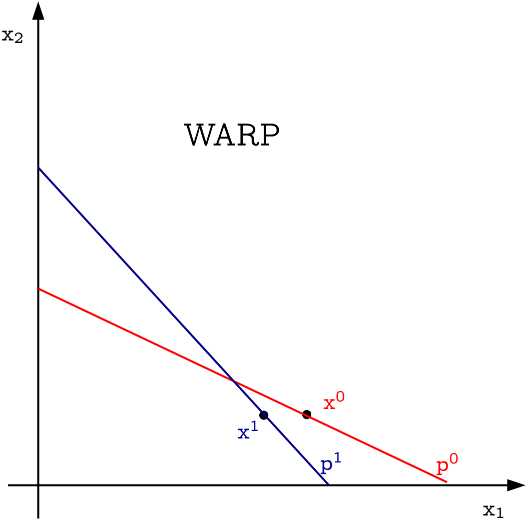
Definition (WARP)

A consumer's choice behavior satisfies WARP if $\forall x^0, x^1$ with x^0 chosen at p^0 and x^1 chosen at p^1

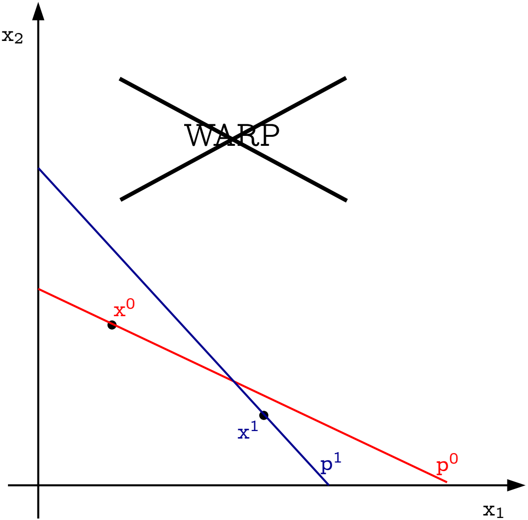
$$p^0 x^1 \leq p^0 x^0 \Rightarrow p^1 x^0 > p^1 x^1$$

WARP holds if whenever x^0 is revealed preferred to x^1, x^1 is never revealed preferred to x^0.

Weak Axiom of Revealed Preferences - WARP



Weak Axiom of Revealed Preferences - WARP



Demand function and WARP

Does the Marshallian demand satisfies the WARP?

Under usual assumptions we know that the solution to the UMP is unique. Suppose that at p^0 and p^1 our consumer chooses

$$x^0 \text{ and } x^1 \text{ and that } p^1 x^1 \leq p^0 x^0 \quad .$$

Then since x^1 is affordable and not chosen $u(x^0) > u(x^1)$. At p^1 on the contrary, x^0 is not chosen so it must be $p^1 x^0 > p^1 x^1$. Hence

$$p^1 x^1 \leq p^0 x^0 \quad \Rightarrow \quad p^1 x^0 > p^1 x^1 \quad \text{[WARP]}$$

WARP and the choice function

Assume:

- consumer's behavior satisfies WARP
- $x(p, y)$ is his choice function (not a demand function!)
- BB: $\forall p > 0 \quad px(p, y) = y$

Under these assumptions the choice function $x(p, y)$ is homogeneous of degree 0 in (p, y)

Proof.

Suppose at p^0 the consumer chooses x^0 and hence $p^0x^0 = y^0$. Analogously at p^1 he chooses x^1 and hence $p^1x^1 = y^1$. Assume $p^1 = tp^0$ and consequently $y^1 = ty^0$. From BB it follows that

$$p^1x^1 = p^0x^0 \Rightarrow \begin{cases} tp^0x^1 = tp^0x^0 \\ p^1x^1 = p^1x^0 \end{cases} \Rightarrow \begin{cases} p^0x^1 = p^0x^0 \\ p^1x^1 = p^1x^0 \end{cases} .$$

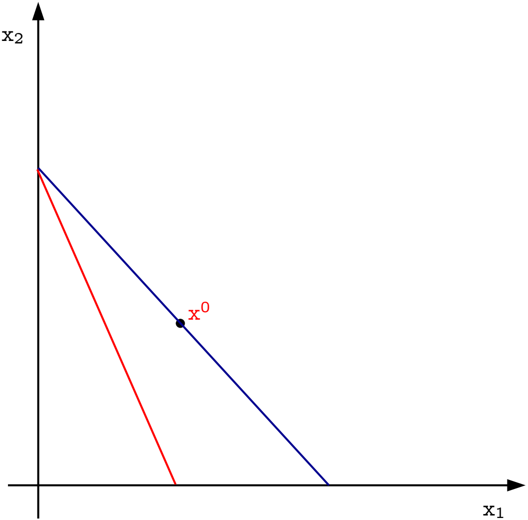
IF $x^0 \neq x^1$ then WARP implies $p^1x^1 < p^1x^0$, contraddiction. Hence, it must be $x^1 = x^0$ but this means that $x(p, y)$ is homogeneous of degree zero.

□

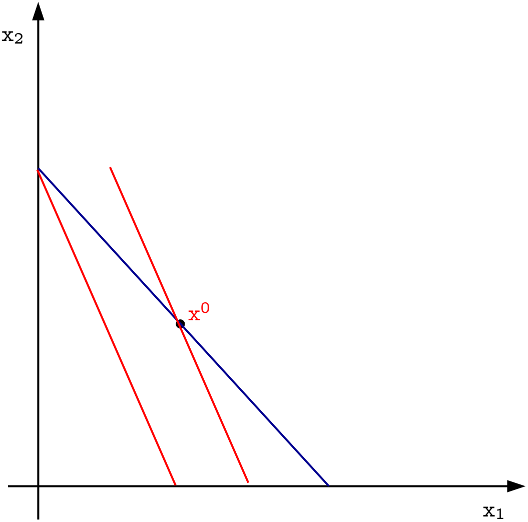
Slutsky compensated demand

The Slutsky compensation when prices vary arbitrarily is obtained by changing the income so that the consumer can just afford the initial bundle.

Slutsky compensated demand



Slutsky compensated demand



Slutsky compensated demand

The Slutsky compensation when prices vary arbitrarily is obtained by changing the income so that the consumer can just afford the initial bundle.

Consequently, under these circumstances, at price p the consumer's income will be px^0 and his the function, describing his choice behavior, $x(p, px^0)$,

Under these assumptions the Slutsky matrix associated with the choice function $x(p, y)$ is negative semidefinite.

Proof.

Consider the following situation

$$p^0 > 0 \quad y^0 > 0 \qquad\qquad p^0 > 0 \quad y^0 > 0$$
$$x^0 = x(p^0, p^0 x^0) \qquad\qquad x^1 = x(p^1, p^1 x^0)$$

$$[\text{WARP}][*] \quad p^0 x^0 \leq p^0 x^1 \text{ since } \begin{cases} x^0 = x^1 \Rightarrow p^0 x^0 = p^0 x^1 \\ x^0 \neq x^1 \Rightarrow p^0 x^0 < p^0 x^1 \end{cases}$$
$$[\text{BB}][*] \quad p^1 x^0 = p^1 x(p^1, p^1 x^0)$$

$$[*] - [**] \quad (p^1 x^0 - p^0 x^0) \leq p^1 x(p^1, p^1 x^0) - p^0 x^1$$
$$(p^1 - p^0) x^0 \geq (p^1 - p^0) x(p^1, p^1 x^0) \qquad \forall p^1$$

Proof. (Cont'ed).

Since the last holds $\forall p^1$, it holds also for $p^1 = p^0 + tz \quad \forall t > 0$ and $\forall x \in \mathbb{R}^n$(negative and positive). This implies

$$(p^0 + tz - p^0)x^0 \geq (p^0 + tz - p^0)x(p^0 + tz, (p^0 + tz)x^0)$$
$$zx^0 \geq zx(p^0 + tz, (p^0 + tz)x^0) \quad .$$

Since $p^0 > 0$, for fixed z we may choose $\bar{t} > 0$ small enough so that $p^0 + tz > 0 \quad \forall t \in [0, \bar{t})$. Then

$$\text{IF } t = 0 \Rightarrow zx^0 = zx(p^0, p^0x^0)$$
$$\text{THEN } f(t) \equiv zx(p^0 + tz, (p^0 + tz)x^0)$$

is maximized on $[0, \bar{t})$ at $t = 0$ and hence $f'(t)|_{t=0} \leq 0$. Let's take the derivative of $f(t)$.

Two goods example

$$
\begin{aligned}
f(t) &\equiv \mathbf{z}x(\mathbf{p^0} + t\mathbf{z}, (\mathbf{p^0} + t\mathbf{z})\mathbf{x^0}) \\
\frac{df(t)}{dt} &= z_1 \left[\frac{\partial x_1}{\partial p_1}\frac{dp_1}{dt} + \frac{\partial x_1}{\partial p_2}\frac{dp_2}{dt} + \frac{\partial x_1}{\partial y}\frac{dy}{dt} \right] + \\
&\left[\frac{\partial x_2}{\partial p_1}\frac{dp_1}{dt} + \frac{\partial x_2}{\partial p_2}\frac{dp_2}{dt} + \frac{\partial x_2}{\partial y}\frac{dy}{dt} \right] \\
\frac{df(t)}{dt} &= z_1 \left[\frac{\partial x_1}{\partial p_1} z_1 + \frac{\partial x_1}{\partial p_2} z_2 + \frac{\partial x_1}{\partial y}(z_1 x_1 + z_2 x_2) \right] + \\
&\left[\frac{\partial x_2}{\partial p_1} z_1 + \frac{\partial x_2}{\partial p_2} z_2 + \frac{\partial x_2}{\partial y}(z_1 x_1 + z_2 x_2) \right] \\
&= z_1 \left[\sum_{j=1}^{2} \frac{\partial x_1}{\partial p_j} z_j + \frac{\partial x_1}{\partial y} z_j x_j \right] + z_2 \left[\sum_{j=1}^{2} \frac{\partial x_2}{\partial p_j} z_j + \frac{\partial x_2}{\partial y} z_j x_j \right] \\
&= \sum_{i=1}^{2} z_i \sum_{j=1}^{2} \frac{\partial x_i}{\partial p_j} z_j + \frac{\partial x_i}{\partial y} x_j z_j \quad .
\end{aligned}
$$

Proof. (Cont'ed).

Hence in the n-goods case

$$\frac{df(t)}{dt} = \sum_{i=1}^{n} z_i \sum_{j=1}^{n} \frac{\partial x_i}{\partial p_j} z_j + \frac{\partial x_i}{\partial y} x_j z_j \leq 0 \quad .$$

Since $z \in \mathbb{R}^n$ is arbitrary this means the matrix whose ij element is

$$\frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial y} x_j$$

is negative semidefinite. But this matrix is precisely the Slutsky matrix associated with the choice function $x(p, y)$.

□

We have just proven the following

Theorem

If a choice function $x(p, y)$ satisfies WARP and BB the it is homogeneous of degree zero in (p, y) and posses a Slustky matrix negative semidefinite.

What is missing to complete the recovery of a full fledged demand function? To assure symmetry WARP is not enough.

Definition (SARP)

SARP is satisfied IF, for any sequence of distinct consumption bundles $x^0, x^1, \ldots, x^k$ where x^0 is revealed preferred to x^1, x^1 to x^2 … x^{k-1} to x^k, THEN it's not the case that x^k is revealed preferred to x^0.