1. DEFINITION

student id	1	2	3	4	5
age (x)	30	22	14	18	24

X_1 represents "age" of student "1"

this sub-script identifies the student

$X_1 = 30$ $X_2 = 22$ $X_3 = 14$ $X_4 = 18$ $X_5 = 24$

In general

X_i represents the age of the generic student "i"

i is a generic name for an index.
IT could have been anything else.

x is a generic name for a variable.
It could have been anything else.

Sometimes we need To sum the age of the 5 students as

$$30 + 22 + 14 + 18 + 24 \qquad (= 108)$$

$$X_1 + X_2 + X_3 + X_4 + X_5 \qquad (= 108)$$

$$\sum_{i=1}^{5} X_i \qquad (= 108)$$

Σ = "sigma" in greek and represents the operator "sum"

$\sum_{1}^{5}{}_i x_i$ must be read as

$\boxed{\text{sum the } x_i \text{ with } i \text{ that goes from } 1 \text{ to } 5}$

ex. $\sum_{4}^{5}{}_i x_i = x_4 + x_5 = 18 + 24 = 42$

$\sum_{1}^{1}{}_i x_i = x_1 = 30$

This notation Turns out To be very useful when we have several students, for example 100 the sum of their ages is simply represented as

$$\sum_{1}^{100}{}_i x_i = x_1 + x_2 + \dots + x_{100}$$

In general with n students

$$\sum_{1}^{n}{}_i x_i = \underbrace{x_1 + x_2 + \dots + x_i + \dots x_n}_{\text{Note the use of } x_i \text{ To represent the "generic" student "}i\text{".}}$$

remark. Think

$\sum_{1}^{n}{}_i x_i = \sum_{1}^{n}{}_j x_j$ TRUE FALSE

$\sum_{1}^{n}{}_i x_i = \sum_{1}^{m}{}_i x_i$ TRUE FALSE

$m < n$

2. PROPERTIES and EXAMPLES

- sum of a constant

$$\sum_{i=1}^{n} 3 = \underbrace{3+3+\dots+3+\dots+3}_{n \text{ Times}} = 3n$$

$$\sum_{i=1}^{n} a = a+a+\dots+a+\dots+a = na$$

(4S) $$\sum_{i=m}^{n} a = \underset{m}{a} + \underset{m+1}{a} + \dots + \underset{n}{a} = (n-m+1)a$$

$m<n$

- sum with product by a constant

(4S) $$\sum_{i=1}^{n} a x_i = a x_1 + a x_2 + \dots + a x_n = a \sum_{i=1}^{n} x_i$$

- sum of sums

(4S) $$\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} (x_i + y_i)$$

$$\sum_{i=1}^{n} (a) + \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} (a + x_i) = na + \sum_{i=1}^{n} x_i$$

- playing with the index

$$\left.\begin{aligned} \sum_{i=1}^{n} i a^{i+1} &= 1a^2 + 2a^3 + \dots + na^{n+1} \\ \sum_{j=2}^{n+1} a^j (j-1) &= 1a^2 + 2a^3 + \dots + n a^{n+1} \end{aligned}\right\} \sum_{i=1}^{n} i a^{i+1} = \sum_{j=2}^{n+1} (j-1) a^j$$

3. DOUBLE SUMS

IT is interesting To generalize to double sums

$$\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} = \sum_{i=1}^{n} \left(x_{i1} + x_{i2} + \dots + x_{im} \right) =$$

$$= \left(x_{11} + x_{12} + \dots + x_{1m}\right) + \left(x_{21} + x_{22} + \dots + x_{2m}\right) + \dots + \left(x_{n1} + x_{n2} + \dots + x_{nm}\right)$$

(4S) $$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} x_i y_j \right) = \sum_{i=1}^{n} \left(x_i \sum_{j=1}^{m} x_j \right) =$$

$$= \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} y_j \right) = \left(\sum_{j=1}^{n} y_j \right) \left(\sum_{i=1}^{n} x_i \right)$$

(4S) $$\left(\sum_{i=1}^{n} x_i \right)^2 = \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{j=1}^{n} x_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j$$

Let's see what happens if $n = 3$

$$\sum_{i=1}^{3} \sum_{j=1}^{3} x_i x_j = \big(x_1 x_1 + x_1 x_2 + x_1 x_3 +$$
$$+ x_2 x_1 + x_2 x_2 + x_2 x_3 +$$
$$+ x_3 x_1 + x_3 x_2 + x_3 x_3 \big)$$

Hence in general

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j = \big(x_1 x_1 + x_1 x_2 + \dots + x_1 x_n +$$
$$x_2 x_1 + x_2 x_2 + \dots + x_2 x_n +$$
$$\vdots$$
$$x_n x_1 + x_n x_2 + \dots + x_n x_n \big)$$

pluggin out the diagonal

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j = \sum_{i=1}^{n} x_i^2 + \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} x_i x_j$$

$$= \left(\sum_{i=1}^{n} x_i \right)^2$$

$$\left(\sum_{i=1}^{n} x_i \right)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j =$$

$$= \sum_{i=1}^{n} x_i^2 + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{n}$$

4. USEFUL SUMS

$$\sum_{1}^{n} i = $$

$$* \quad \underline{(i+1)^2 - i^2} = i^2 + 1 + 2i - i^2 = \underline{2i+1}$$

$$* \quad \sum_{1}^{n} (i+1)^2 - i^2 = \sum_{1}^{n} (2i+1)$$

Then

$$\sum_{1}^{n} (i+1)^2 - \sum_{1}^{n} i^2 =$$

$$\to 2^2 + 3^2 + \dots + (n+1)^2$$

$$\to -1^2 - 2^2 - \dots - n^2 =$$

$$= (n+1)^2 - 1$$

and

$$(n+1)^2 - 1 = \sum_{1}^{n} (2i+1)$$

$$(n+1)^2 - 1 = 2\sum_{1}^{n} i + \sum_{1}^{n} 1$$

$$n^2 + 1 + 2n - 1 = 2\sum_{1}^{n} i + n$$

$$2\sum_{1}^{n} i = n^2 + 2n - n$$

$$\sum_{1}^{n} i = \frac{n(n+1)}{2}$$

45 $\quad \sum_{1}^{n} i^2 =$

$$* \quad (i+1)^3 - i^3 = 3i^2 + 3i + 1$$

[...]

$$\sum_{1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$