EXERCISES on PROBABILITY.

(1) List the possible outcomes of tossing two coins

TT HT TH HH

Since they have the same probability of being observed then this probability must be $\frac{1}{4} = 0.25$ since $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} = 1$.

Define Y = # of "heads" observed. The possible outcomes of Y are

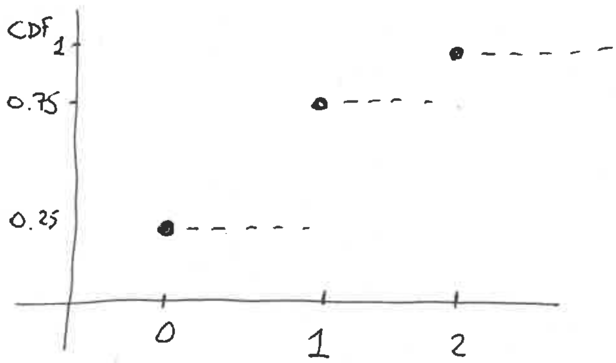
0	1	2
TT	HT TH	HH

then the PDF of Y is

y_i	$P_r(Y=y_i)$
0	$1/4 = 0.25$
1	$1/2 = 0.5$
2	$1/4 = 0.25$

The CDF of Y is

y_i	$CDF(y_i)$
0	0.25
1	0.75
2	1



2. $E[Y] = \sum_1^3 {}_i\, y_i P_i = (0.25)\,0 + (0.5)\,1 + (0.25)\,2 = 1$

$$VAR[Y] = \sum_1^3 {}_i (y_i - E[Y])^2 p_i = 0.25(0-1)^2 + 0.5(1-1)^2 + 0.25(2-1)^2 =$$

$$= 0.25 + 0.25 = 0.5 = \sigma_Y^2$$

$$SKW[Y] = \frac{\sum_1^3 {}_i (y_i - E[Y])^3}{\sigma_Y^3} = \frac{0.25(0-1)^3 + 0.5(1-1)^3 + 0.25(2-1)^3}{(\sqrt{\sigma_Y^2})^3} =$$

$$= \frac{-0.25 + 0.25}{(\sqrt{\sigma_Y^2})^3} = 0 \quad [\text{Simmetry}]$$

$$KUR[Y] = \frac{\sum_1^3 {}_i (y_i - E[Y])^4}{\sigma_Y^4} = \frac{0.25(0-1)^4 + 0.5(1-1)^4 + 0.25(2-1)^4}{(0.5)^2}$$

$$= \frac{0.5}{0.25} = 2$$

3. $Y = 2000 + 0.2\,(X)$

$E[Y] = E[2000 + 0.2\,X] = 2000 + 0.2\,E[X] = 2000 + 0.2\,\mu_X$

$VAR[Y] = 0.04\,\sigma_X^2$

4.

	X: $x=0$ R	X: $x=1$ NR	$P(y)$
Y: LT $y=50$	0.15	0.07	0.22
Y: $y=10$ ST	0.15	0.63	0.78
$P(x)$	0.30	0.7	$\boxed{1}$

$$E[Y] = \sum_{i=1}^{2} y_i P_i = 0.22(50) + 0.78(10) = 18.8$$

$$E[X] = \sum_{i=1}^{2} x_i P_i = 0.30(0) + 0.7(1) = 0.7$$

$$P_2(Y=50 \mid X=0) = \frac{0.15}{0.15+0.15} = 0.5$$

$$P_2(Y=10 \mid X=0) = \frac{0.15}{0.15+0.15} = 0.5$$

$$E[Y \mid X=0] = \sum_{i=1}^{2} y_i P(Y_i \mid X=0) = 50(0.5) + 10(0.5) = 30$$

$$E[Y \mid X=1] = \sum_{i=1}^{2} y_i P(Y=y_i \mid X=1) = 50(0.1) + 10(0.9) = 14$$

[redacted] $= P(X=0)\,E[Y \mid X=0] + P(X=1)\,E[Y \mid X=1]$

$$= (0.3) \times (0.3) + (0.7) \times (14) = 18.8$$

5. $X = \begin{cases} 0 & Pr = 0.1 \\ 1 & Pr = 0.9 \end{cases}$ $\quad Y \sim N(0,4) \quad W \sim N(0.16)$

Let $S = (X)Y + (1-X)W$

$E[Y^2] = \sigma_Y^2 + \mu_Y^2 = 4 + 0 = 4$

$E[W^2] = \sigma_W^2 + \mu_W^2 = 16 + 0 = 16$

$E[Y^3] = SKEW[Y] - \mu_Y^3 = SKEW[Y] = 0$ since the Normal distribution is symmetric. Similarly for $E[W^3]$.

The LIE states $E[Y] = E[E[Y|X]]$ for any X, Y.

$E[S|X=0] = E[W] = 0$

$E[S|X=1] = E[Y] = 0$

$\Rightarrow E[S] = 0.9\, E[Y|X=1] + 0.1\, E[Y|X=0]$
$= 0$

$E[S^2|X=0] = E[W^2] = 16$

$E[S^2|X=1] = E[Y^2] = 4$

$\Rightarrow E[S^2] = 0.9 * 4 + 0.1 * 16 = 5.2$

$E[S^3|X=0] = E[W^3] = 0$

$E[S^3|X=1] = E[Y^3] = 0$

$\Rightarrow E[S^3] = 0$

6. $\bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i \qquad E[\bar{Y}] = \mu_Y = 10 \qquad VAR[\bar{Y}] = \frac{\sigma_Y^2}{n} = \frac{4}{\sqrt{n}}$

$$SD[\bar{Y}] = \frac{\sigma_Y}{\sqrt{n}} = \frac{2}{\sqrt{n}}$$

The CLT states that, under regularity conditions, when n is large

$$\frac{\bar{Y} - E[\bar{Y}]}{SD[\bar{Y}]} \sim N(0,1)$$

that is

$$\frac{\bar{Y} - 10}{2/\sqrt{n}} \sim N(0,1)$$

$n = 20 \qquad Pr(9.6 \leq \bar{Y} \leq 10.4) = Pr\left(\frac{9.6-10}{2/\sqrt{20}} \leq \frac{\bar{Y}-10}{2/\sqrt{20}} \leq \frac{10.4-10}{2/\sqrt{20}} \right)$

$$= Pr(-0.89 \leq Z \leq 0.89) =$$

$$= Pr(Z \leq 0.89) - (1 - Pr(Z \leq 0.89)) =$$

$$= 0.8133 - (1 - 0.8133) = 0.6266 \approx 0.63$$

$n = 100 \qquad Pr(9.6 \leq \bar{Y} \leq 10.4) = [\ldots] = 0.95$

$n = 1000 \qquad Pr(9.6 \leq \bar{Y} \leq 10.4) = [\ldots] \approx 1$

7.

$$Y = \begin{cases} 1 & 0.6 \\ 0 & 0.4 \end{cases} \qquad \begin{matrix} y_i & Pr(Y=y_i) \end{matrix}$$

Again we can use the CLT. However first we compute

$$E[Y] = \sum_{1}^{2} y_i P_i = 0.6 * 1 + 0.4 * 0 = 0.6$$

$$VAR[Y] = \sum_{1}^{2} (y_i - E[Y])^2 P_i = 0.6 (1-0.6)^2 + 0.4 (0-0.6)^2 =$$

$$= 0.6 \times (0.4)^2 + 0.4 * (-0.6)^2 = 0.24$$

The CLT states

$$\frac{\bar{Y} - 0.6}{\sqrt{\frac{0.24}{n}}} \sim N(0,1)$$

$$Pr(\bar{Y} > 0.64) = Pr\left(\frac{\bar{Y} - 0.6}{\sqrt{0.24/n}} > \frac{0.64 - 0.6}{\left(\frac{0.24}{\sqrt{n}}\right)} \right)$$

$$= Pr\left(Z > \frac{0.04}{\sqrt{\frac{0.24}{50}}} \right) = Pr(Z > 0.58) =$$

$$= 1 - Pr(Z \le 0.58) = 1 - 0.7190 = 0.281$$

We want to find a n such that $Pr(0.56 \le \bar{Y} \le 0.64) = 0.95$

First we use statistical Tables to find A in

$Pr(-A \le \bar{Y} \le A) = 0.95 \Rightarrow A = 1.96$ [check Table on your TextBook]

Next using again the CLT

$$Pr(0.56 \le \bar{Y} \le 0.64) = Pr\left(-\frac{0.04}{\sqrt{\frac{0.24}{n}}} \le Z \le \frac{0.04}{\sqrt{\frac{0.24}{n}}} \right) \Rightarrow 0.95$$

$$\frac{0.04}{\sqrt{\frac{0.24}{n}}} = 1.96 \Rightarrow n \approx 576$$