

Microeconomics

Angelo Secchi

PSME
Academic Year 2011/12

■ Angelo Secchi

- Université Paris 1 Panthéon-Sorbonne, Centre d'Economie
- Bocconi University (undergrad. economics), Sant'Anna School of Advanced Studies (PhD)
- industrial organization, industrial dynamics and applied econometrics

■ Senne Jung

- PhD candidate Paris School of Economics and Université Paris 1 Panthéon-Sorbonne
- Yonsei University (undergrad. mathematics), Seoul National University (MA in Economics), Paris school of economics (MA in Economics)
- applied microeconomics, labour economics, behavioral economics

1 / 173

2 / 173

Organization

Everything [Microeconomics EPI web page](#)

■ Readings

- no required textbook
- Jehle and Reny - Advanced Microeconomic Theory (expensive)
- Varian - Microeconomic Analyses
- further readings and Syllabus on EPI
- slides available Monday evenings on EPI, print them out before classes
- warnings on slides: howto use them, handle them with care

■ Contacts

- angelo.secchi@univ-paris1.fr Put **[MICRO]** in the Subject
- in case you need to talk with me, drop an email to make an appointment

3 / 173

Organization - cont'ed

■ Exam: written, closed book

- MIDTERM November 3,2011 2pm-5pm
- FINAL January 10,2012 2pm-5pm

■ Grading policy

- 50% midterm and 50% final
- grades range: 0 – 20. There is no formal Failure threshold.
- To validate the semester you must have an average grade of 10. If your semester average is lower than 10, you must retake all the exams with a grade lower than 10.
- To validate the year and hence to obtain the Master you must have an overall average grade of 10.

■ Code of Conduct: honesty, integrity and respect.

- ignorance is not an excuse, check the [Code on the EPI](#)
- Board of Examiners is very strict in applying the Code

4 / 173

Organization - cont'ed

■ Class timing(tentative)

2.00-3.30	3.30-3.45	3.45-4.45	4.45-5.00
class	s break	class	R&S Q&A

■ Maths

Maths is not the subject of this course but the formalization of economic intuitions discussed in ECON101 is the main aim of our classes. If you have problems contact one of your tutors.

■ (Hopefully useless) Notes

- Exams will not be rescheduled. No exceptions. Do not make plans or travel reservations for the exam week. This is not a valid excuse for rescheduling exams.
- In case of exceptional events contact, as soon as possible, one of your tutors.
- Please turn off cell phones and other noisemakers during class.

5 / 173

Introduction

preferences
utility
choice

Consumer Theory

preference relations
utility function

Quick Maths Refresh

The Consumer Problem

indirect utility function
expenditure function
the Slutsky equation

Revealed Preferences

Uncertainty

Tentative schedule

20	Sep	Introduction.
27	Sep	Consumer theory.
4	Oct	Duality, Slutsky and the integrability problem.
11	Oct	Production theory.
18	Oct	Uncertainty.
25	Oct	Partial equilibrium analysis. Catch up and review before the midterm exam.
3	Nov	MIDTERM EXAM (covers classes until 18 Oct)
8	Nov	General equilibrium in pure exchange economy.
15	Nov	General equilibrium with production. Core.
22	Nov	Game theory I: strategic form games.
29	Nov	Game theory II: extensive form games.
6	Dec	Adverse selection, moral hazard and the agency problem.
13	Dec	Catch up and review before the final exam.

6 / 173

Introduction

preferences
utility
choice

Consumer Theory

preference relations
utility function

Quick Maths Refresh

The Consumer Problem

indirect utility function
expenditure function
the Slutsky equation

Revealed Preferences

Uncertainty

7 / 173

8 / 173

Ariel Rubinstein's Warnings

Many economists have strong and conflicting views about what economic theory is

- set of theories that can/should be tested
- bag of tools to be used by economic agents
- framework through which professional and academic economists view the world

Microeconomics

Microeconomics is a collection of models in which the primitives are details about the behavior of units called economic agents (not necessarily individuals).

An economic agent is described in our models as a unit, that following a deliberation process (rational choice), make a choice from a set of available alternatives. The process consists in

- asking himself “What is desirable?”
- asking himself “What is feasible?”
- choosing the most desirable from among the feasible alternatives.

Note the **order of the stages**: rational economic agent's desires are independent of the set of alternatives. Also rationality in economics does not contain judgments about desires.

9 / 173

Ariel Rubinstein's Warnings

Disappointment for those of you who have come to this course with practical motivations

- economic theory is no more than an arena for the investigation of concepts we use in thinking about economics in real life and models provide a language to try to understand reality better BUT
- I do not view economic models as an attempt to describe the world or to provide tools for predicting the future
- I object to looking for an ultimate truth in economic theory
- I do not expect it to be the foundation for any policy recommendation

However, an economic model differs substantially from a purely **mathematical model** in that it is a combination of a mathematical model and its **interpretation**.

10 / 173

Microeconomics

Economists have become increasingly aware that almost all people, almost all the time, do not practice this kind of deliberation!

Is it meaningful to talk about the concept of “being good” even in a society where all people are evil? Similarly . . .

Further reading

Rubinstein, A., “Dilemmas of an economic theorist”
<http://arielrubinstein.tau.ac.il/papers/74.pdf>[clickable]

11 / 173

12 / 173

Introduction

preferences

utility

choice

Consumer Theory

preference relations

utility function

Quick Maths Refresh

The Consumer Problem

indirect utility function

expenditure function

the Slutsky equation

Revealed Preferences

Uncertainty

Preferences

Why do we start with preferences?

“Preferences” are one of the building block of microeconomic models

- imagine you want to fully describe the preferences of an economic agent over a set X of alternatives
- what should this description include?
- is there any requirement this system of preferences should fulfill?

Let's introduce this concept in a simple **intuitive manner** with a modeling exercise.

13 / 173

14 / 173

Preferences - Questionnaire Q

- consider an economic agent and a set of alternatives X
- for all x and y in X the agent is confronted with the question $Q(x, y)$: **how do you compare x and y ?**
- select one of the following
 - I prefer x to y , $x \succ y$
 - I prefer y to x , $x \prec y$
 - I'm indifferent, $x \sim y$

15 / 173

Preferences - Legal answer

- one and only one choice is permitted
- we exclude answers such as: they are incomparable, I do not know, I prefer x to y but also y to x , it depends on my girlfriend's taste or on the weather
- we ignore the intensity of preferences

Do we qualify all the legal answers to the questionnaire Q qualify as preferences over the set X ? NO

16 / 173

Preferences - Consistency restrictions

We adopt two “consistency” restrictions

- $Q(x, y)$ must be equal to $Q(y, x)$ [No order effect]
- the answers to $Q(x, y)$, $Q(y, z)$ and $Q(x, z)$ must be consistent. That is, if the answer to $Q(x, y)$, $Q(y, z)$ is “preference” then also $Q(x, z)$ must be answered “preference”. Same in case of indifference. [Transitive]

Preferences - Legal answer

- one and only one choice is permitted
- $R(x, y)$ must be equal to $R(y, x)$ [No order effect]
- if the answers to $R(x, y)$ and $R(y, z)$ is “yes” then also the answer to $R(x, z)$ must be “yes” [Transitive]

The two representations of agent’s preferences are equivalent

Preferences - Questionnaire R

A second way to think about preferences is via a different questionnaire. Again

- consider an economic agent and a set of alternatives X
- for all x and y in X the agent is confronted with the question $R(x, y)$: **is x at least as preferred as y ?**
- select one of the following
 - Yes
 - No

17 / 173

Preferences - caveats and limitations

Transitivity seems a reasonable assumption. However,

- real experiments show frequent violations of transitivity
- it excludes individuals that base their judgment on procedures such
 - aggregation of primitive considerations
 - similarity among alternatives

18 / 173

19 / 173

20 / 173

Aggregation of primitive considerations may break transitivity

Suppose an individual has to choose among 3 alternatives on which he has three primitive considerations and finds an alternative better than another one if the majority of primitives support it.

Consider the following rankings table

PRIMITIVES	x	y	z
consid. A	1	2	3
consid. B	3	1	2
consid. C	2	3	1

In this case $x \succ y$, $y \succ z$ but $z \succ x$

21 / 173

Introduction

preferences
utility
choice

Consumer Theory

preference relations
utility function

Quick Maths Refresh

The Consumer Problem

indirect utility function
expenditure function
the Slutsky equation

Revealed Preferences

Uncertainty

23 / 173

Similarity among alternatives may break transitivity

Consider an individual whose attitude is “the larger the better”. However he cannot distinguish among alternatives unless their difference is greater than 1.

$$f(x, y) = x \text{ if } x \geq y + 1$$

$$f(x, y) = I \text{ if } |x - y| < 1$$

then $1.5 \sim 0.8$, $0.8 \sim 0.3$ but $1.5 \sim 0.3$ is not true

22 / 173

Utility

Think of examples of preferences.

- if the alternatives are few we may simply rank them:
base additive colors(RGB): red \succ green \succ blue
- if alternatives are many (most often)
basketball players: I prefer the taller basket players

All these examples can naturally be specified by a statement of the form

$$x \succ y \text{ if } V(x) \geq V(y)$$

where $V : X \rightarrow \mathbb{R}$ is a function that attaches a real number to each element in the set of alternatives X . $V(x)$ is called **utility function** and \succeq is said to have an **utility representation**

24 / 173

Utility

Comments

- Note that the statement $x \succ y$ if $V(x) \geq V(y)$ always defines a preference relation since the relation \geq on \mathbb{R} satisfies completeness and transitivity
- It is possible to avoid the notion of a utility representation and to do economics only with the notion of preferences
- when defining a preference relation using a utility function, the function has an intuitive meaning that carries with it additional information
- in contrast, when the utility function is formed to represent an existing preference relation, the utility function per se has no meaning.

25 / 173

Introduction

preferences
utility
choice

Consumer Theory

preference relations
utility function

Quick Maths Refresh

The Consumer Problem

indirect utility function
expenditure function
the Slutsky equation

Revealed Preferences

Uncertainty

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- in contrast, when the utility function is formed to represent an existing preference relation, the utility function per se has no meaning. **Absolute values are meaningless!**

26 / 173

Choice

- until now we have avoided any reference to agent's behavior, we only talked about preferences as a summary of the decision maker's mental attitude toward a set of alternatives
- consider now the usual set of alternatives X , a **choice problem** may be viewed as a nonempty subset of X , and we refer to a **choice from $A \subseteq X$ as specifying one of A 's members**
- representing a choice problem in this way implies that agent's choice does not depend on the way alternatives are presented
 - ignores the order in which alternatives are presented
 - ignores the number of time alternatives appear in A
 - ignores if they are "defaults"

27 / 173

28 / 173

Choice - Rationality assumption

It is typically assumed in economics that choice is an outcome of rational deliberation

economic agent has in mind a preference relation \succeq on the set X and, given any choice problem $A \subseteq X$, he chooses $x \in A$ OPTIMAL with respect to \succeq

29 / 173

Choice - Critiques to rationality

FRAMING

Consider the following formulations of a plan to react to a national pandemic

- formulation A
 - (a) 400 people will die
 - (b) 0 will die with probability 1/3 and 600 will die with probability 2/3

- formulation B
 - (c) 200 people will be saved
 - (d) 600 will be saved with probability 1/3 and 0 will be saved with probability 2/3

31 / 173

Choice - Critiques to rationality

- Economists were often criticized because of the assumption of rationality
- economists' defense: we don't really need this assumption. All we need to assume is that the decision maker's behavior can be described **as if** he were maximizing some preference relation
- a modern attack, behavioral economics. Economics+Psychology, A. Tversky and D. Kahneman (Nobel laureate in economics in 2002). They have demonstrated not only that rationality is often **violated**, but that there are **systematic reasons** for the violations. . .

30 / 173

Choice - Critiques to rationality

The two questions were presented to 2 groups of people: 78% chose b and only 28% chose d

Both questions presented in the above order to game theory students: 78% chose b and only 49% chose d .

- large proportion of subjects gave different answers to the two problems
- results highlight the sensitivity of choices to the framing of alternatives
- what is more damaging for rational decision making than observing different answer to the same question only because they are presented differently?

32 / 173

SIMILARITIES

Consider the following gambling games

	color	white	red	green	yellow
(a)	probability	90	6	1	3
	price €	0	45	30	-15

	color	white	red	green	yellow
(b)	probability	90	7	1	2
	price €	0	45	-10	-15

Which one would you choose?

- experiments on two groups of students: in the first group 58% selected '(a)' while in the second group nobody chose '(c)'
- when both choice problems, one after the other, were posed to the same students: 52% chose '(a)' and 7% '(c)'
- rationale: we often transfer complicated problem into simpler ones by “canceling” similar elements

Consider now the following gambling games

	color	white	red	green	blue	yellow
(c)	probability	90	6	1	1	2
	price €	0	45	30	-15	-15

	color	white	red	green	blue	yellow
(d)	probability	90	6	1	1	2
	price €	0	45	45	-10	-15

Which one would you choose?

REASON-BASED CHOICE

- example of cameras: one group with two alternatives (same brand, one lower quality for 170€ and the other higher quality 240€), another group with a third alternative (top quality camera for 470€). Guess what happens to the number of people choosing the 240€ camera. . .
- making choice sometimes requires finding motives to pick one alternative over the others
- in this kind of situations, choice is based on “internal reasons”, and it is often difficult to reconcile the actual choices with the rational paradigm

REASON-BASED CHOICE

- example of cameras: one group with two alternatives (same brand, one lower quality for 170€ and the other higher quality 240€), another group with a third alternative (top quality camera for 470€). Guess what happens to the number of people choosing the 240€ camera. . . **Internal reason: compromising alternative**
- making choice sometimes requires finding motives to pick one alternative over the others
- in this kind of situations, choice is based on “internal reasons”, and it is often difficult to reconcile the actual choices with the rational paradigm

- preferences on X are binary relations on X satisfying **completeness** and **transitivity**. They can be formalized in different but equivalent ways
- a system of preferences can be represented by an **utility function** which has no meaning other than that of representing a preference relation
- a choice problem is seen as a nonempty subset of X and a choice consists in selecting the element of this subset that is **optimal** wrt the system of preferences considered

37 / 173

38 / 173

Introduction

preferences
utility
choice

Consumer Theory

preference relations
utility function

Quick Maths Refresh

The Consumer Problem

indirect utility function
expenditure function
the Slutsky equation

Revealed Preferences

Uncertainty

Consumer Theory - Introduction

Consumer theory is characterized **axiomatically**: few “meaningful” assumptions are set forth; the rest of the theory is then developed via the **process of deduction**

There are four main building blocks:

- CONSUMPTION SET: set of all the alternatives X ;
- FEASIBLE SET: set of all the conceivable and realistically obtainable alternatives $B \subset X$ given the consumers’ circumstances (income, prices);
- PREFERENCE RELATION: specifies limits, forms of consistencies and inconsistencies of the consumer’s ability to make choice among alternatives;
- BEHAVIORAL ASSUMPTION: principles guiding the consumer in his choice.

39 / 173

40 / 173

Introduction

preferences
utility
choice

Consumer Theory

preference relations
utility function

Quick Maths Refresh

The Consumer Problem

indirect utility function
expenditure function
the Slutsky equation

Revealed Preferences

Uncertainty

Preference relations

- we represent the consumer's preference by a binary relation \succeq defined on the consumption set X .

$$x^1 \succeq x^2 \quad x^1 \text{ is at least as good as } x^2$$

where $x^i = (x_1, \dots, x_n)$ is **consumption bundle** with $x \in X \subseteq \mathbb{R}_+^n$

- using a binary relation implies that the consumer examines only two alternative consumption bundles at a time and makes a decision between those two
- we impose two other important restrictions

41 / 173

Preference - Axioms

COMPLETENESS: For all x^1 and $x^2 \in X$, either $x^1 \succeq x^2$ or $x^2 \succeq x^1$

The consumer has always the ability to discriminate and the necessary knowledge to evaluate alternatives. Philosophically questionable: extreme situations.

TRANSITIVITY: For all x^1, x^2 and $x^3 \in X$ if $x^1 \succeq x^2$ and $x^2 \succeq x^3$ then $x^1 \succeq x^3$

As discussed, this axiom imposes a very peculiar form of consistencies to consumers' choices

43 / 173

42 / 173

Preference - Axioms

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44 / 173

Preferences - Definitions

PREFERENCE RELATION: a binary relation \succeq defined on X is called a preference relation if it satisfies **completeness** and **transitivity**

STRICT PREFERENCE: a binary relation \succ defined on X as

$$x^1 \succ x^2 \text{ IFF } x^1 \succeq x^2 \text{ and } x^2 \not\succeq x^1$$

is called strict preference relation induced by \succeq

INDIFFERENCE: a binary relation \sim defined on X as

$$x^1 \sim x^2 \text{ IFF } x^1 \succeq x^2 \text{ and } x^2 \succeq x^1$$

is called indifference relation induced by \succeq

Are SPR and IR transitive? Are they complete?

45 / 173

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47 / 173

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Are SPR and IR transitive? Are they complete?

46 / 173

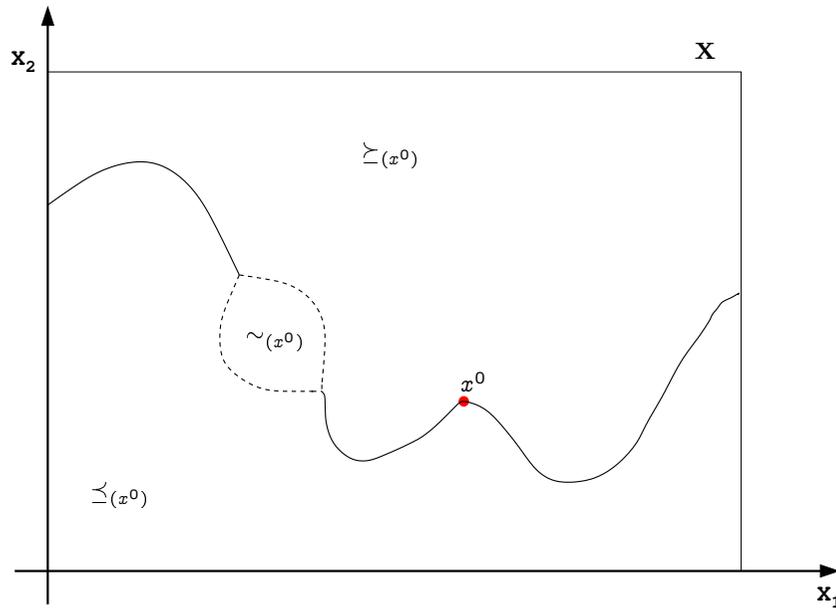
Preferences - Preference set induced by \succeq

Let $x^0 \in X$. Then we can define

- $\succeq_{(x^0)} \equiv \{x \mid x \in X, x \succeq x^0\}$ “at least as good” set
- $\preceq_{(x^0)} \equiv \{x \mid x \in X, x \preceq x^0\}$ “no better than” set
- $\succ_{(x^0)} \equiv \{x \mid x \in X, x \succ x^0\}$ “preferred to” set
- $\prec_{(x^0)} \equiv \{x \mid x \in X, x \prec x^0\}$ “worse than” set
- $\sim_{(x^0)} \equiv \{x \mid x \in X, x \sim x^0\}$ “indifference” set

48 / 173

Preferences - Preference set induced by \succeq



49 / 173

Preferences - Axioms

CONTINUITY: For all $x \in X$, both $\succeq(x)$ and $\preceq(x)$ are closed sets.

- continuity guarantees that sudden preference reversals do not occur; continuity can be restated as
if each element y_n of a sequence is $\succeq x$ and $\lim_{n \rightarrow \infty} y_n = y$
then
 $y \succeq x$
- consequence: if both $\succeq(x)$ and $\preceq(x)$ are closed then also $\sim(x)$ is closed, non open area exists.

Proof.

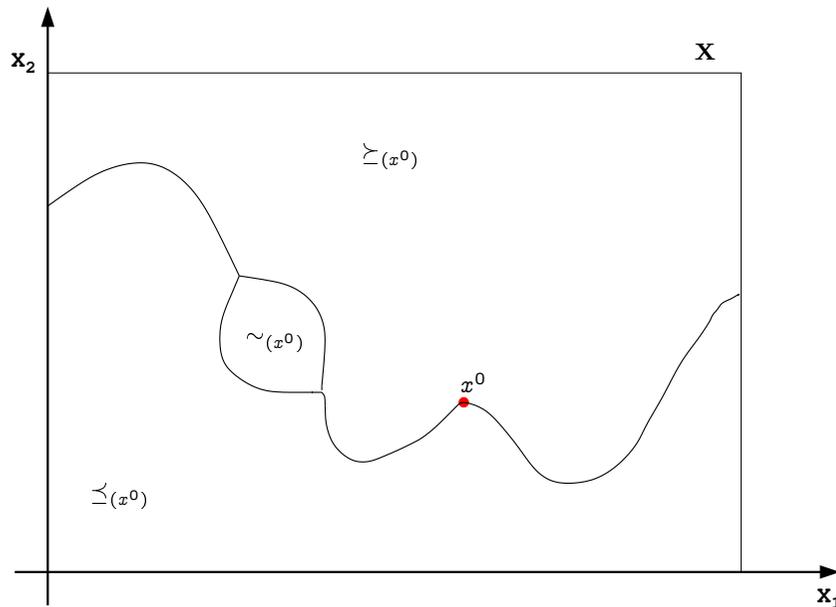
Let A and B two closed sets. Then A^c and B^c are open.

Unions of open sets are open sets.

De Morgan's Law states that $A^c \cup B^c = (A \cap B)^c$ hence $(A \cap B)^c$ is open and $(A \cap B)$ is closed

50 / 173

Preferences - Axioms



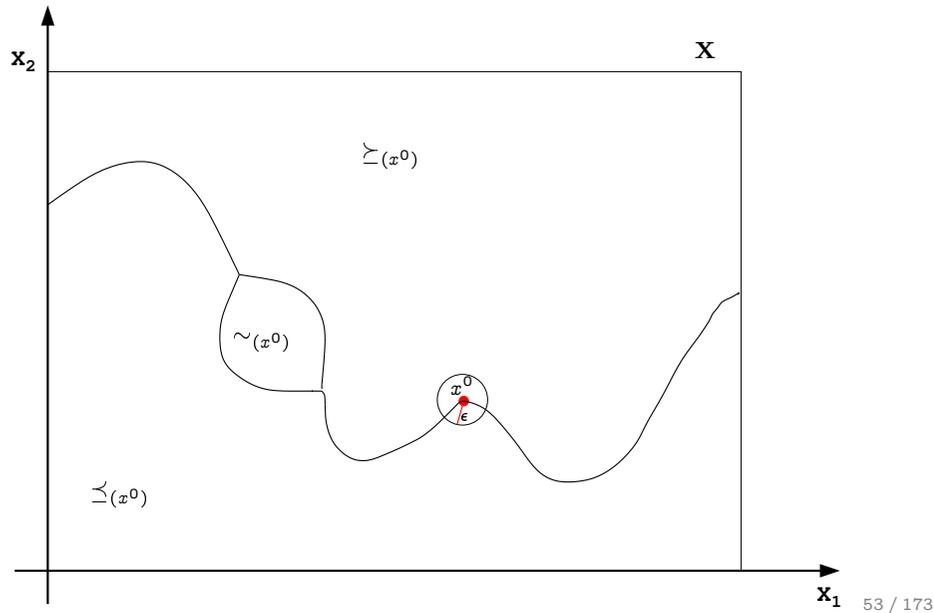
51 / 173

LOCAL NONSATIATION: For all $x^0 \in X$ and for all $\epsilon > 0$, there exists some $x \in B_\epsilon(x^0) \cap \mathbb{R}_+^n$ such that $x \succ x^0$

52 / 173

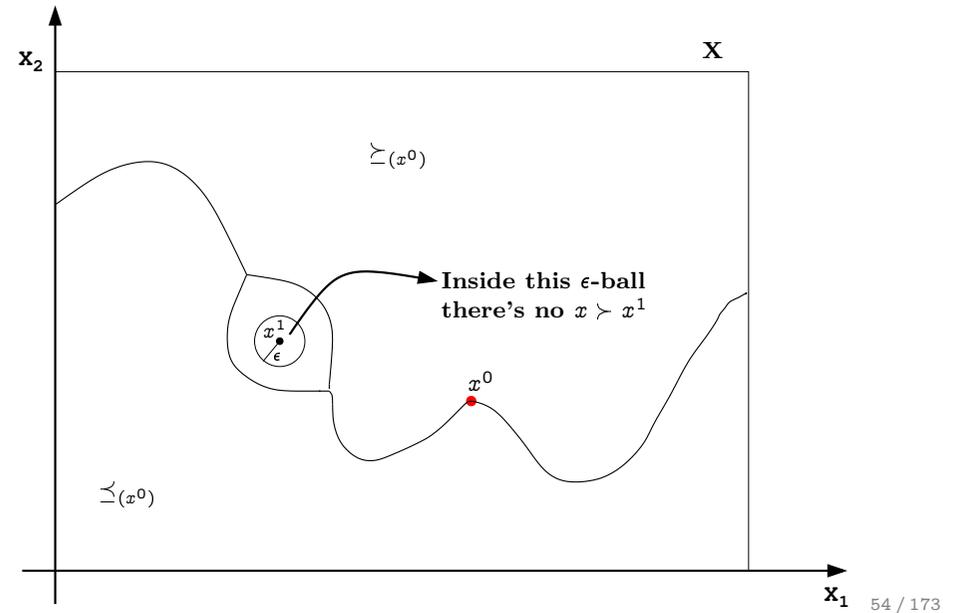
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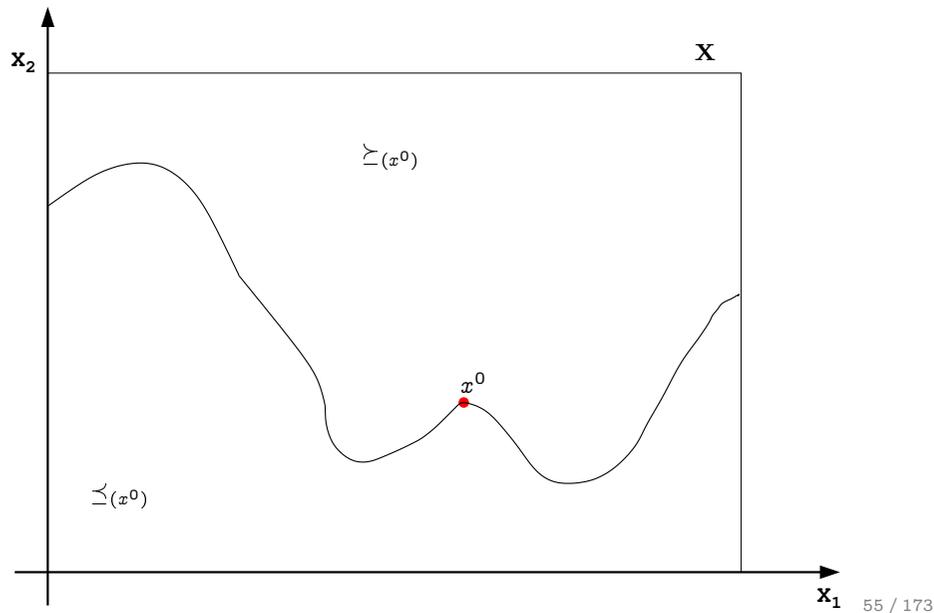
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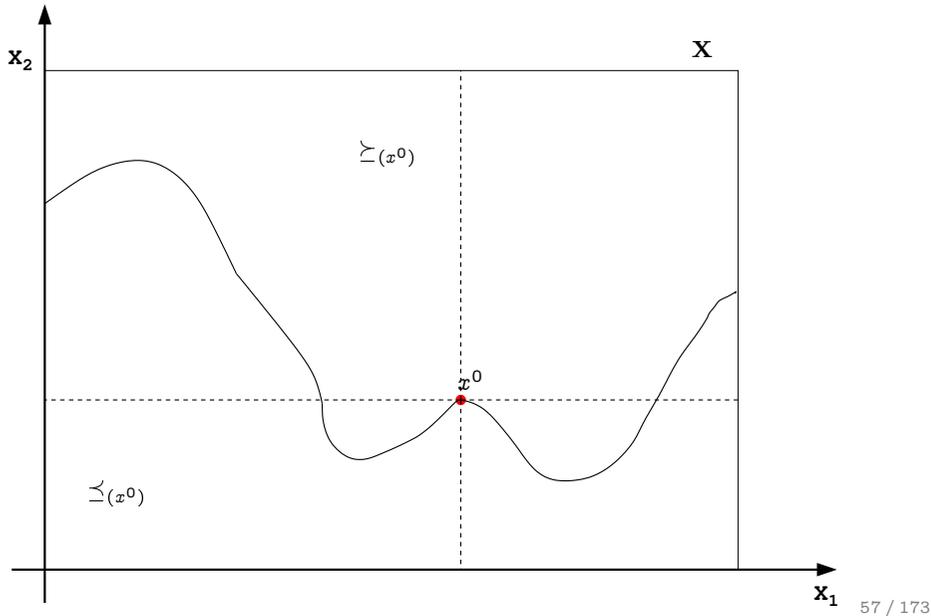


Preferences - Axioms

STRICT MONOTONICITY: For all x^0 and $x^1 \in \mathbb{R}_+^n$ if $x^0 \geq x^1$ then $x^0 \succeq x^1$, while if $x^0 > x^1$ then $x^0 \succ x^1$

Preferences - Axioms

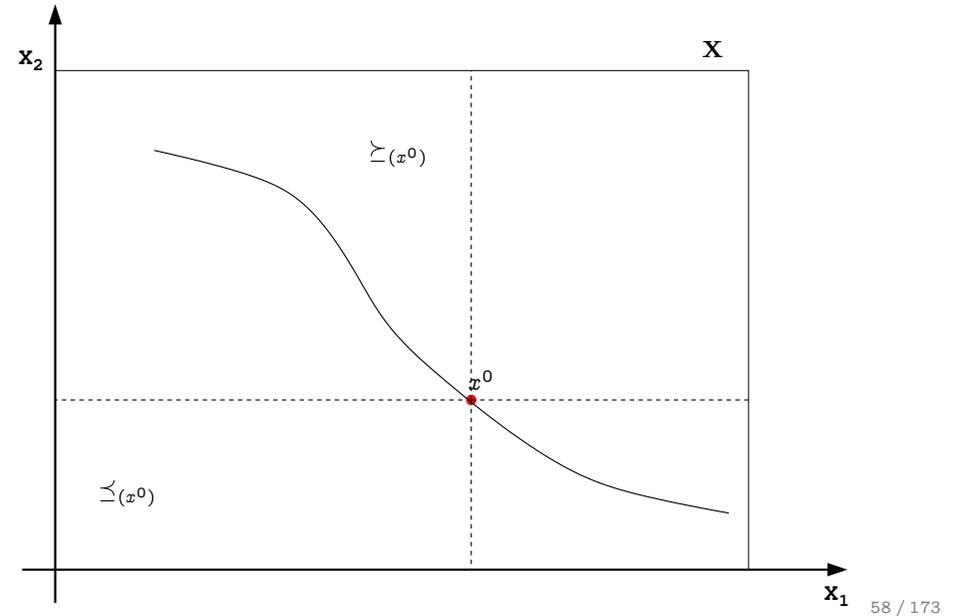
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57 / 173

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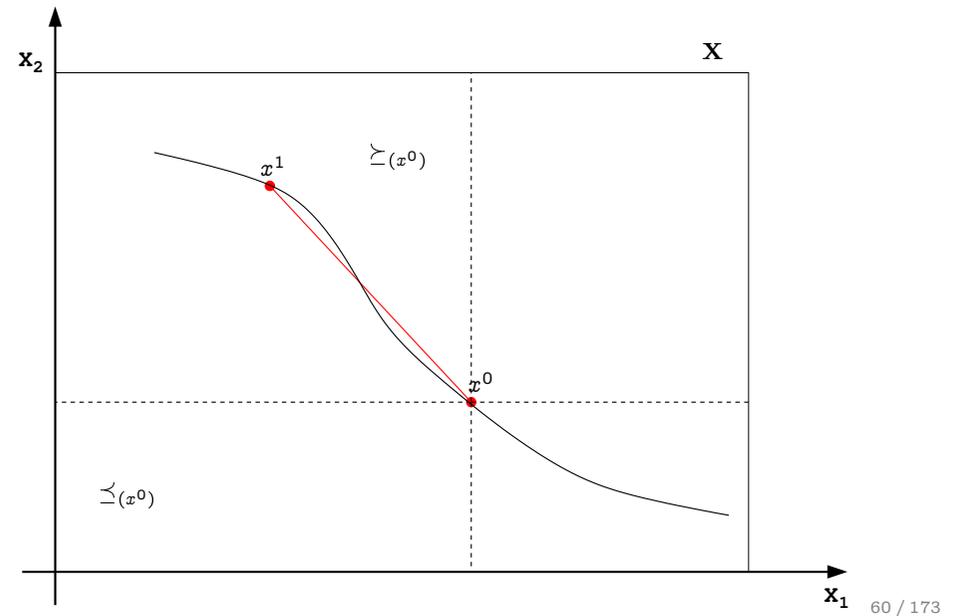
58 / 173

Preferences - Axioms

STRICT CONVEXITY: If $x^0 \neq x^1$ and $x^1 \succeq x^0$ then $tx^1 + (1-t)x^0 \succ x^0 \forall t \in (0, 1)$

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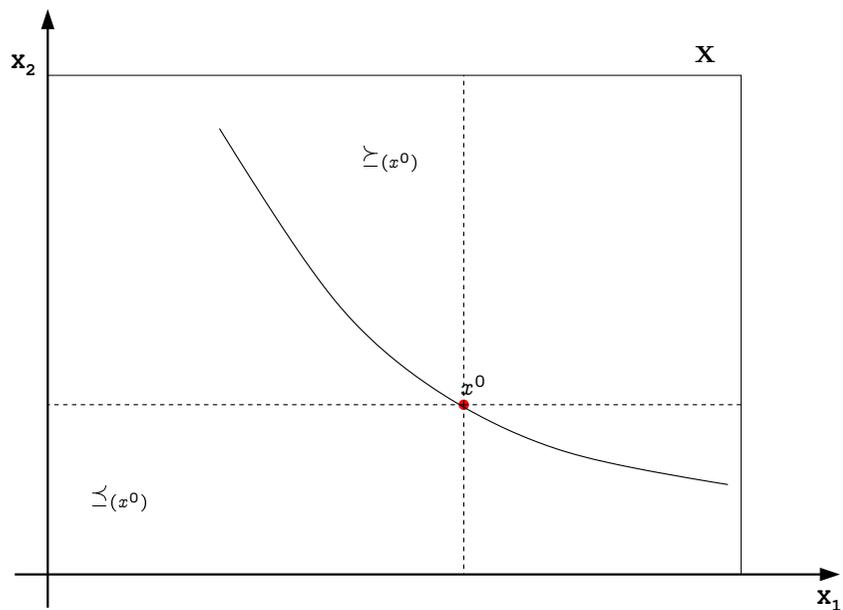


59 / 173

60 / 173

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61 / 173

Preferences axioms - Sum up

- Completeness and Transitivity describe a consumer who can make **consistent comparisons** among alternatives
- Continuity guarantees the existence of **topologically nice** “ \succeq ” and “ \succ ” sets
- Strict Monotonicity and Strict Convexity require that tastes display some form of **“more is better”** and **“more balanced is better”**

63 / 173

Preferences - Axioms

STRICT CONVEXITY: If $x^0 \neq x^1$ and $x^1 \succeq x^0$ then $tx^1 + (1-t)x^0 \succ x^0 \forall t \in (0, 1)$

Intuitively strict convexity can be interpreted in two ways:

- more balanced consumption bundles are preferred
- the rate at which the consumer is willing to give up x_2 in exchange of x_1 (MRS) decreases with x_1

62 / 173

Introduction

preferences
utility
choice

Consumer Theory

preference relations
utility function

Quick Maths Refresh

The Consumer Problem

indirect utility function
expenditure function
the Slutsky equation

Revealed Preferences

Uncertainty

64 / 173

Utility function

In modern economic theory an utility function is a convenient way to summarize the info contained in the consumer's preference relation.

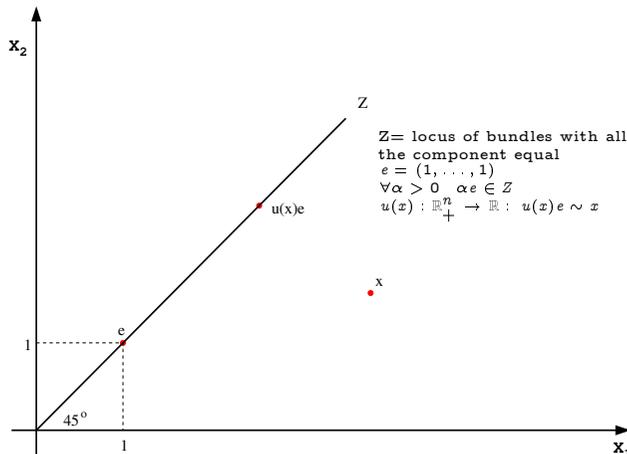
UTILITY FUNCTION: a real valued function $u : \mathbb{R}_+^n \rightarrow \mathbb{R}$ is called utility function representing the preference relation \succeq if

$$\forall x^0, x^1 \in \mathbb{R}_+^n \quad u(x^0) \geq u(x^1) \Leftrightarrow x^0 \succeq x^1$$

Utility function

Theorem

If \succeq is complete, transitive, continuous and strictly monotonic there exists a continuous real valued function $u : \mathbb{R}_+^n \rightarrow \mathbb{R}$ which represents \succeq .



Utility function

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Proof.

1. Does the number $u(x)$ satisfying its definition exist?
2. If yes, is it uniquely determined, so that $u(x)$ is a well-defined function?

Let us define

$$A \equiv \{t \geq 0 \mid te \succeq x\}$$

$$B \equiv \{t \geq 0 \mid te \preceq x\}$$

If $t^* \in A \cap B$ then $t^*e \sim x$ and hence $u(x) = t^*$ satisfies our definition. Is $A \cap B$ a non-empty set? **Continuity** of \succeq implies that both A and B are closed set, then by **strict monotonicity**

$$t \in A \Rightarrow t' \in A \quad \forall t' \geq t \Rightarrow A = [\bar{t}, +\infty)$$

$$t \in B \Rightarrow t' \in B \quad \forall t' \leq t \Rightarrow B = [0, \underline{t}]$$

Proof. - cont'ed.

Since \succeq is **complete** it is either $te \succeq x$ or $te \preceq x$ and then

$t \in A \cup B$. Finally, since $A \cup B = \mathbb{R}_+$ then

$A \cup B = [0, \underline{t}] \cup [\bar{t}, +\infty)$ from which it follows that $\underline{t} \leq \bar{t}$ and hence $A \cap B \neq \emptyset$.

We prove the uniqueness by contradiction. Suppose there are two numbers satisfying our definition, $u_1(x) \neq u_2(x)$, then

$$u_1(x) = t_1 e \sim x$$

$$u_2(x) = t_2 e \sim x$$

which implies (by **transitivity**) $t_1 e \sim t_2 e$ and hence $t_1 = t_2$ that is in contradiction with our assumption.

Then $\forall x \in \mathbb{R}_+^n \exists! u(x) : u(x)e \sim x$. The proof is concluded by showing that $u(x)$ is continuous and represents \succeq .

Utility function

- This theorem is clearly very important since allows us to avoid(!) the primitive set-theoretic preference relation.
- Is the utility function $u(x)$ representing \succeq unique?

Theorem

Let \succeq be a preference relation on \mathbb{R}_+^n and suppose $u(x)$ represent it. Then $v(x) = f(u(x))$ represents \succeq IFF $f(\cdot)$ is strictly increasing on the set of values taken by $u(x)$

Proof. - cont'ed.

To this aim, let's consider two consumption bundles x^1 and x^2

$$\begin{aligned}
 x^1 \succeq x^2 &\Leftrightarrow u(x^1)e \sim x^1 \succeq x^2 \sim u(x^2)e \\
 &\Leftrightarrow u(x^1)e \succeq u(x^2)e \Leftrightarrow u(x^1) \geq u(x^2)
 \end{aligned}$$

To prove that $u(x)$ is continuous we need to show that the inverse image under $u(\cdot)$ of every open ball (a, b) in \mathbb{R} is open in \mathbb{R}_+^n . This means

$$\begin{aligned}
 u^{-1}((a, b)) &= \{x \in \mathbb{R}_+^n \mid a < u(x) < b\} \\
 &= \{x \in \mathbb{R}_+^n \mid ae \prec x \prec be\} \\
 &= \succ_{(ae)} \cap \prec_{(be)}
 \end{aligned}$$

By continuity of \succeq both $\prec_{(ae)}$ and $\succ_{(be)}$ are closed, then their complements are open. By De Morgan's Law and since the intersection of open set is an open set it follows that $u(x)$ is continuous.

Introduction

- preferences
- utility
- choice

Consumer Theory

- preference relations
- utility function

Quick Maths Refresh

The Consumer Problem

- indirect utility function
- expenditure function
- the Slutsky equation

Revealed Preferences

Uncertainty

Maths refresh

Geometric interpretation of the gradient. Consider a function $u(x) : \mathbb{R}^2 \Rightarrow \mathbb{R}$. Using the chain rule we can compute the rate of change of $u(x_1, x_2)$ at any given point x^* in any given direction $v = (v_1, v_2)$

$$\begin{aligned}x &= x^* + tv \quad t \in \mathbb{R} \\g(t) &\equiv u(x^* + tv) = u(x_1^* + tv_1, x_2^* + tv_2) \\g'(0) &= \left(\frac{\partial u}{\partial x_1}(x^*) \quad \frac{\partial u}{\partial x_2}(x^*) \right) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = Du(x^*) v\end{aligned}$$

Sometimes $Du(x^*)$ is written as a column vector and is called **gradient**, $\nabla u(x^*)$. Using the **dot product**

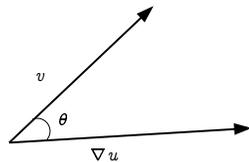
$$Du(x^*)v = \nabla u(x^*) \cdot v = \sum_{i=1}^2 \frac{\partial u}{\partial x_i}(x^*) v_i = \|\nabla u(x^*)\| \|v\| \cos \theta$$

73 / 173

Maths refresh

In what direction does the function $u(x)$ increase most rapidly?

$$\nabla u(x^*) \cdot v = \|\nabla u(x^*)\| \|v\| \cos \theta$$



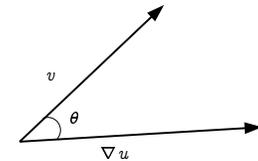
Maths refresh

In what direction does the function $u(x)$ increase most rapidly?

Maths refresh

In what direction does the function $u(x)$ increase most rapidly?

$$\nabla u(x^*) \cdot v = \|\nabla u(x^*)\| \|v\| \cos \theta$$



Clearly $\nabla u(x^*) \cdot v$ is maximized when $\cos \theta = 1$, that is when $\theta = 0^\circ$

Theorem

Let $u(x) : D \rightarrow \mathbb{R}$ be a C^1 function. At any $x \in D$ for which $\nabla u(x) \neq 0$ the gradient $\nabla u(x)$ points into the direction in which $u(x)$ increases most rapidly.

75 / 173

74 / 173

76 / 173

Theorem (Dini's)

Let $u(x_1, x_2)$ be a C^1 function on a ball about $(x_1^0, x_2^0) \in \mathbb{R}^2$.
 Suppose that $u(x_1^0, x_2^0) = c$ and consider the level curve

$$u(x_1, x_2) = c \quad .$$

If $\partial u / \partial x_2 \neq 0$, there exists a function $x_2 = x_2(x_1)$ defined in an interval I around x_1^0 such that

$$\begin{aligned} u(x_1, x_2(x_1)) &\equiv c \quad \forall x \in I \\ x_2(x_1^0) &= x_2^0 \\ x_2'(x_1^0) &= -\frac{\frac{\partial u}{\partial x_1}(x_1^0, x_2^0)}{\frac{\partial u}{\partial x_2}(x_1^0, x_2^0)} \end{aligned}$$

Proof.(sketch).

In general the directional derivative is given by

$$Du(x^*)v = \nabla u(x^*) \cdot v \quad .$$

If we want to move along the level curve it must be

$$\begin{pmatrix} \frac{\partial u}{\partial x_1}(x_1^0, x_2^0) \\ \frac{\partial u}{\partial x_2}(x_1^0, x_2^0) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -\frac{\frac{\partial u}{\partial x_1}(x_1^0, x_2^0)}{\frac{\partial u}{\partial x_2}(x_1^0, x_2^0)} \end{pmatrix} = 0$$

and since

$$\nabla u \cdot v = \|\nabla u(x)\| \|v\| \cos \theta \tag{1}$$

we get that θ , the angle between the gradient and the tangent to the level curve must be equal to 90° .

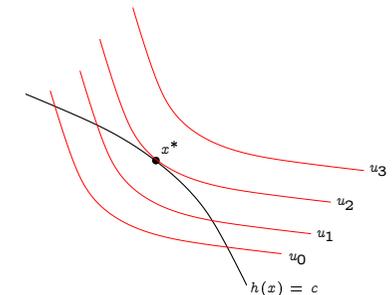
Theorem (Gradient and level curves)

Let $u(x_1, x_2)$ be a C^1 function on a ball about $(x_1^0, x_2^0) \in \mathbb{R}^2$.
 Suppose that (x_1^0, x_2^0) is a regular point of u (either $\partial u / \partial x_1(x_1^0, x_2^0) \neq 0$ or $\partial u / \partial x_2(x_1^0, x_2^0) \neq 0$). Then

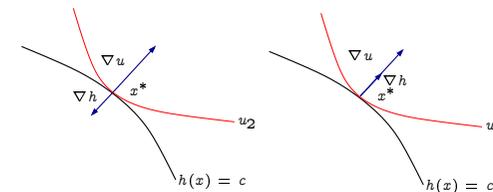
$$\nabla u(x_1^0, x_2^0) \perp u(x_1^0, x_2^0) = c \quad .$$

Geometric intuition behind the Lagrangian. Suppose to have the following problem

$$\begin{aligned} \text{MAX}_x \quad &u(x) \\ \text{s.t.} \quad &h(x) = c \end{aligned}$$



In x^* the level curves and the constraint present the same slope, hence



Maths refresh

In either cases the two gradients are scalar multiples of each other. If we call the multiplier λ^* we get

$$\nabla u(x^*) = \lambda^* \nabla h(x^*)$$

that is

$$\begin{cases} \frac{\partial u}{\partial x_1}(x^*) = \lambda^* \frac{\partial h}{\partial x_1}(x^*) \\ \frac{\partial u}{\partial x_2}(x^*) = \lambda^* \frac{\partial h}{\partial x_2}(x^*) \end{cases}$$

which together with the constraint $h(x) = c$ represent the FOC of our problem. These conditions can be obtained by defining the Lagrangian $L = u(x) - \lambda(h(x) - c)$ and requiring that

$$\begin{cases} \frac{\partial L}{\partial x_1}(x^*) = 0 \\ \frac{\partial L}{\partial x_2}(x^*) = 0 \\ \frac{\partial L}{\partial \lambda}(x^*) = 0 \end{cases}$$

Maths refresh

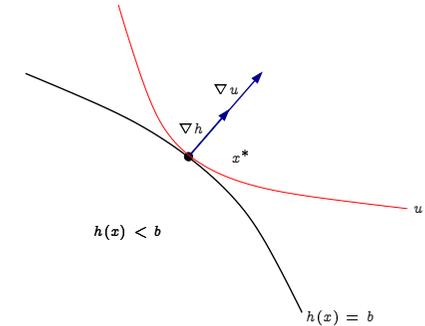
Hence with the inequality constraint we must add to the usual condition a positivity constraint on λ

$$\begin{cases} \nabla u(x^*) = \lambda^* \nabla h(x^*) \\ \lambda^* \geq 0 \end{cases}$$

Maths refresh

Inequality constraint. Suppose to have the following problem

$$\begin{aligned} \text{MAX}_x \quad & u(x) \\ \text{s.t.} \quad & h(x) \leq b \end{aligned}$$



If $h(x^*) = b$ again x^* the level curve and the constraint present the same slope. However,

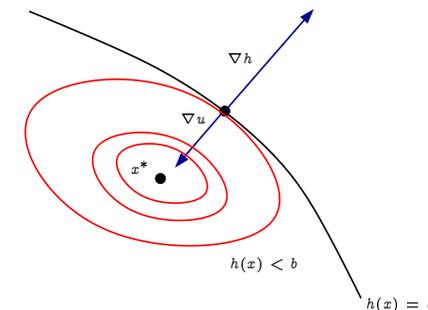
1. since the constraint is $h(x) \leq b$ we know that $\nabla h(x)$ must point to the region where $h(x) > b$
2. since x^* must maximize $u(x)$ we are sure that $\nabla u(x)$ must point to the region where $h(x) > b$

81 / 173

82 / 173

Maths refresh

With the inequality constraint we must investigate also what happens if in $h(x^*) < b$. In this case



- $\nabla u(x^*)$ and $\nabla h(x^*)$ point in opposite directions and λ^* in $\nabla u(x^*) = \lambda^* \nabla h(x^*)$ must be negative
- x^* is an unconstrained maxima so the usual FOC $\nabla u(x^*) = 0$ apply

83 / 173

84 / 173

Lagrangian - Sum up.

$$\begin{aligned} \text{MAX}_{x_1, x_2} \quad & u(x_1, x_2) \\ \text{s.t.} \quad & h(x_1, x_2) \leq b \end{aligned} \quad \begin{cases} \frac{\partial L}{\partial x_1}(x^*) = 0 \\ \frac{\partial L}{\partial x_2}(x^*) = 0 \\ \frac{\partial L}{\partial \lambda}(x^*) = 0 \\ \lambda^*(h(x_1, x_2) - b) = 0 \\ \lambda^* \geq 0 \end{cases}$$

where $L = u(x_1, x_2) - \lambda(h(x_1, x_2) - b)$

Proof.

From the definition of L we get

$$\frac{\partial L}{\partial a}(x^*(a), \lambda^*, a) = \frac{\partial u}{\partial a}(x^*(a); a) - \lambda \frac{\partial h}{\partial a}(x^*(a); a) .$$

Next consider the following

$$\frac{du}{da}(x^*(a), a) = \sum_{i=1}^n \frac{\partial u}{\partial x_i}(x^*(a); a) \frac{\partial x_i}{\partial a} + \frac{\partial u}{\partial a}(x^*(a); a) \cdot 1$$

$$h(x^*(a); a) = 0 \Rightarrow \sum_{i=1}^n \frac{\partial h}{\partial x_i}(x^*(a); a) \frac{\partial x_i}{\partial a} + \frac{\partial h}{\partial a}(x^*(a); a) = 0$$

$$\frac{\partial u}{\partial x_i}(x^*(a); a) = \lambda \frac{\partial h}{\partial x_i}(x^*(a); a) \quad \text{FOC of the Max problem}$$

Envelope Theorem. Let $u(x; a)$ and $h(x; a)$ be $C^1 \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ functions. Let $x^*(a)$ the solution of

$$\begin{aligned} \text{MAX}_x \quad & u(x) \\ \text{s.t.} \quad & h(x; a) = 0 . \end{aligned}$$

Suppose $x^*(a)$ and $\lambda^*(a)$ are C^1 functions of a , then

$$\frac{d}{da} u(x^*(a); a) = \frac{\partial L}{\partial a}(x^*(a), \lambda^*(a); a)$$

where L is the Lagrangian for this problem.

Proof. - Cont'ed.

Substituting the last two conditions in the definition of du/da we get

$$\frac{du}{da}(x^*(a), a) = \underbrace{\lambda \frac{\partial h}{\partial a}(x^*(a); a) + \frac{\partial u}{\partial a}(x^*(a); a)}_{\frac{\partial L}{\partial a}(x^*(a), \lambda^*(a); a)} .$$

□

Introduction

preferences
utility
choice

Consumer Theory

preference relations
utility function

Quick Maths Refresh

The Consumer Problem

indirect utility function
expenditure function
the Slutsky equation

Revealed Preferences

Uncertainty

At the most abstract level we view the consumer as having

1. consumption set \mathbb{R}_+^n
2. preference relation \succsim defined on \mathbb{R}_+^n
3. circumstances (income and prices) that limit the alternatives the consumer is able to achieve thus defining a feasible set $B \subset \mathbb{R}_+^n$
4. motivation to obtain the most preferred alternative according to his preferences

Assumption.

The consumer's preference relation \succsim is complete, transitive, continuous, strictly monotonic and strictly convex in \mathbb{R}_+^n . Therefore \succsim can be represented by a real valued utility function, $u(x)$, that is continuous, strictly increasing and strictly quasiconcave on \mathbb{R}_+^n .

89 / 173

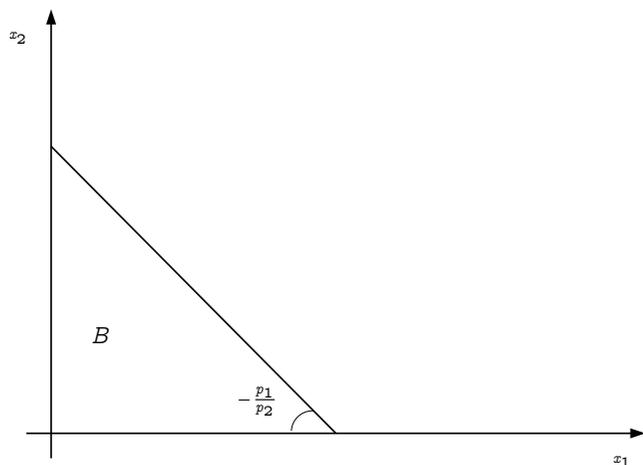
90 / 173

Budget set

Our consumer operates in a **market economy**:

1. there are $p_i \quad \forall i$
2. agents cannot influence prices
3. agents are endowed with an income $y > 0$

Budget set: $B = \{x | x \in \mathbb{R}_+^n; p \cdot x \leq y\}$



91 / 173

Consumer problem: mathematical structure

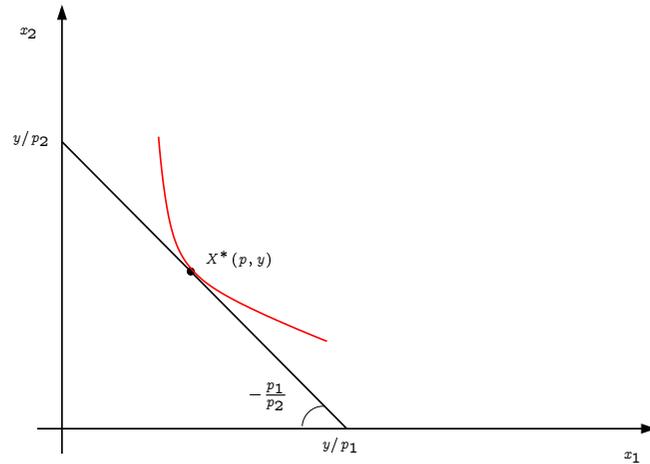
$$\begin{aligned} \text{MAX}_x \quad & u(x) \\ \text{s.t.} \quad & p \cdot x - y \leq 0 \end{aligned}$$

- $u(x)$ is a continuous real valued function
- B is a non empty, closed, bounded (thus compact) subset of \mathbb{R}^n
- $u(x)$ is quasiconcave and B is convex \Rightarrow there exists a unique max (Weierstrass's Theorem)
- preference are strictly monotonic $\Rightarrow x^*$ satisfies the budget constraint with equality

92 / 173

Consumer problem: mathematical structure

$$\begin{aligned} \text{MAX}_x \quad & u(x) \\ \text{s.t.} \quad & p \cdot x - y \leq 0 \end{aligned}$$



Marshallian demand function

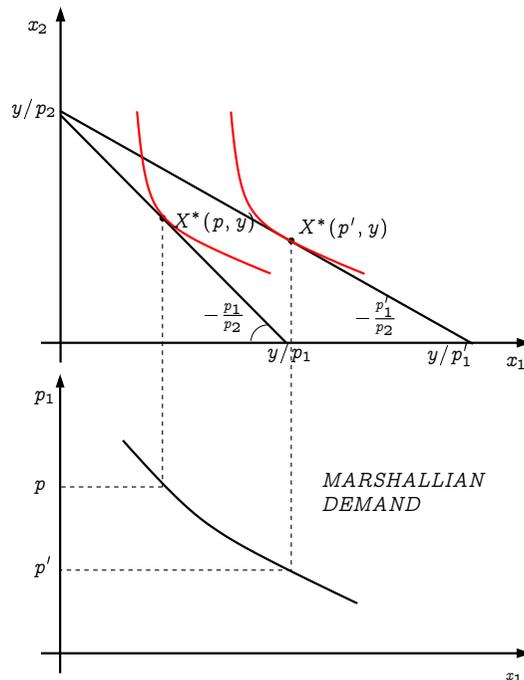
The unique solution to the CP (Consumer Problem) depends on p and y ; hence x^* can be viewed as a function of these parameters

$$x^* = x(p, y) \quad \text{Marshallian demand}$$

93 / 173

94 / 173

Marshallian demand function



CP with differentiable utility

$$\begin{aligned} \text{MAX}_x \quad & u(x) \\ \text{s.t.} \quad & p \cdot x - y \leq 0 \end{aligned}$$

If $u(x)$ is differentiable we can define the Lagrangian

$$L = u(x) - \lambda(px - y)$$

and set the usual FOC

$$\begin{cases} \frac{\partial L}{\partial x_i}(x^*) = \frac{\partial u}{\partial x_i} - \lambda^* p_i = 0 \quad \forall i \\ \frac{\partial L}{\partial \lambda}(x^*) = px - y = 0 \\ \lambda^*(px^* - y) = 0 \quad [\text{redundant}] \\ \lambda^* \geq 0 \end{cases}$$

95 / 173

96 / 173

CP with differentiable utility

- since $\nabla u(x) > 0$ then from the FOC

$$\frac{\partial u}{\partial x_j}(x^*) = \lambda^* p_j \quad \forall j$$

at the equilibrium the marginal utility is proportional to price for all goods

- alternatively,

$$MRS_{jk} = -\frac{\partial u / \partial x_j}{\partial u / \partial x_k} = -\frac{p_j}{p_k} \quad \forall j, k$$

the slope of the indifference curve through x^* is equal to the slope of the budget constraint, where MRS_{jk} is the rate at which x_j can be substituted for x_k with no change in consumer's utility

97 / 173

Lemma

Let $u(x)$ be a C^1 function in \mathbb{R} . Then $u(x)$ is concave in $I \subset \mathbb{R}$ IFF

$$u(x^1) - u(x^0) \leq u'(x^0)(x^1 - x^0) \quad \forall x^1, x^0 \in I$$

This condition generalizes in

$$\nabla u(x^0)(x^1 - x^0) \geq 0 \quad \text{if} \quad u(x^1) \geq u(x^0).$$

Geometric intuition.

CP - Sufficiency condition

In general the FOC are only necessary conditions. In this particular case however

Theorem

If $u(x)$ is continuous and (quasi)concave on \mathbb{R}_+^n and $(p, y) > 0$ then if $u(x)$ is differentiable at x^* and $(x^*, \lambda^*) > 0$ solves

$$\frac{\partial u(x^*)}{\partial x_i} - \lambda^* p_i = 0 \quad \forall i$$

then x^* solves the CP.

To prove this important theorem we need a preliminary result.

98 / 173

Proof.

ONLY IF (sufficient condition). If $u(x)$ is concave then

$$tu(x^1) + (1-t)u(x^0) \leq u(tx^1 + (1-t)x^0)$$

$$u(x^1) - u(x^0) \leq \frac{u(tx^1 + x^0 - tx^0) - u(x^0)}{t(x^1 - x^0)}(x^1 - x^0)$$

$$u(x^1) - u(x^0) \leq \lim_{t \rightarrow 0} \frac{u(x^0 + t(x^1 - x^0)) - u(x^0)}{t(x^1 - x^0)}(x^1 - x^0)$$

$$u(x^1) - u(x^0) \leq u'(x^0)(x^1 - x^0)$$

99 / 173

100 / 173

Proof.

IF (necessary condition). If

$$u(x) - u(y) \leq u'(y)(x - y) \quad \forall x, y \in I \quad \text{then}$$

$$u(x^1) - u((1-t)x^1 + tx^0) \leq tu'((1-t)x^1 + tx^0)(x^1 - x^0) \quad [*]$$

$$u(x^0) - u((1-t)x^1 + tx^0) \leq -(1-t)u'((1-t)x^1 + tx^0)(x^1 - x^0) \quad [**]$$

Multiplying [*] and [**] respectively by $1-t$ and t

$$(1-t)(u(x^1) - u((1-t)x^1 + tx^0)) \leq (1-t)t u'((1-t)x^1 + tx^0)(x^1 - x^0)$$

$$t(u(x^0) - u((1-t)x^1 + tx^0)) \leq -t(1-t) u'((1-t)x^1 + tx^0)(x^1 - x^0)$$

summing up the two equations

$$(1-t)(u(x^1)) + t(u(x^0)) - u((1-t)x^1 + tx^0) \leq 0$$

$$(1-t)(u(x^1)) + t(u(x^0)) \leq u((1-t)x^1 + tx^0)$$

□ 1 / 173

Introduction

preferences

utility

choice

Consumer Theory

preference relations

utility function

Quick Maths Refresh

The Consumer Problem

indirect utility function

expenditure function

the Slutsky equation

Revealed Preferences

Uncertainty

Proof of the sufficiency theorem.

Let's prove the statement by contradiction. Assume x^* such that $\nabla u(x^*) = \lambda^* p$ and with $px = y$ exists. If it is not the solution of the CP then, $\forall t \in [0, 1]$ it must exist an x^0 such that

$$\begin{cases} u(x^0) > u(x^*) \\ px^0 - y \leq 0 \end{cases} \xrightarrow{\text{continuity of } u(x)} \begin{cases} u(tx^0) > u(x^*) \\ p(tx^0) - y < 0 \end{cases}$$

Setting $x^1 = tx^0$ one gets

$$\begin{aligned} \nabla u(x^*)(x^1 - x^*) &= \lambda^* p(x^1 - x^*) \\ &= \lambda^* (px^1 - px^*) \\ &< \lambda^* (y - y) = 0 \end{aligned}$$

which, by the previous Lemma, is in contradiction with the assumption of concavity of $u(x)$.

□

102 / 173

Indirect utility function

The value of $u(x^*)$ reached in the solution of the CP depends on the level of prices p and on the consumer's income y . The relations among p , y and $u(x^*)$ can be summarized by

$$v : \mathbb{R}_+^n \times \mathbb{R}_+ \rightarrow \mathbb{R} \quad v(p, y) = \text{Max}_{x \in \mathbb{R}_+^n} u(x) \quad \text{s.t. } px - y \leq 0 \quad .$$

$v(p, y)$ is called **indirect utility function** and under usual regularity conditions

$$v(p, y) = u(x^m(p, y)) \quad ,$$

where $x^m(p, y)$ is a well defined Marshallian demand function.

Theorem (Properties of $v(p, y)$)

If $u(x)$ is continuous, strictly increasing and differentiable and $(p, y) > 0$ the the associated $v(p, y)$ is

1. continuous on $\mathbb{R}_+^n \times \mathbb{R}_+$
2. homogeneous of degree 0 in (p, y)
3. strictly increasing in y
4. decreasing in p
5. Roy's Identity:

$$x_i(p^0, y^0) = -\frac{\partial v(p^0, y^0)/\partial p_i}{\partial v(p^0, y^0)/\partial y}$$

Proof (sketch).

1. Theorem of the maximum states (approximately) that if $u(x)$ and the constraints in the CP are continuous in the parameters and their domain is compact then $v(p, y)$ is continuous

2. Homogeneity of degree 0 means $v(p, y) = v(tp, ty) \quad \forall t > 0$.
Easy: does the budget constraint vary?

3. From the Envelope theorem

$$\frac{dv(y, p)}{dy} = \frac{\partial L(p, y, \lambda)}{\partial y} = \lambda > 0$$

4. Similarly

$$\frac{dv(y, p)}{dp_i} = \frac{\partial L(p, y, \lambda)}{\partial p_i} = -\lambda x_i < 0$$

5. From 3. and 4. we get directly Roy's Identity.

105 / 173

106 / 173

Introduction

preferences
utility
choice

Consumer Theory

preference relations
utility function

Quick Maths Refresh

The Consumer Problem

indirect utility function
expenditure function
the Slutsky equation

Revealed Preferences

Uncertainty

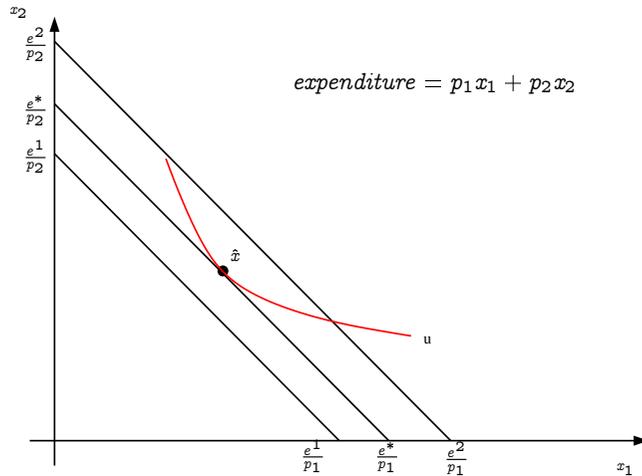
Expenditure function

The expenditure function originates fixing the prices of goods and asking at those prices what's the minimum level of money expenditure the consumer must make to achieve a given level of utility.

107 / 173

108 / 173

Expenditure function



Expenditure function

The expenditure function $e(p, u)$ is defined as

$$e : \mathbb{R}_+^n \times \mathcal{U} \rightarrow \mathbb{R}_+ \quad e(p, u) \equiv \text{Min}_{x \in \mathbb{R}_+^n} px \quad \text{s.t. } u(x) \geq u \quad .$$

109 / 173

110 / 173

Remarks on $e(p, u)$

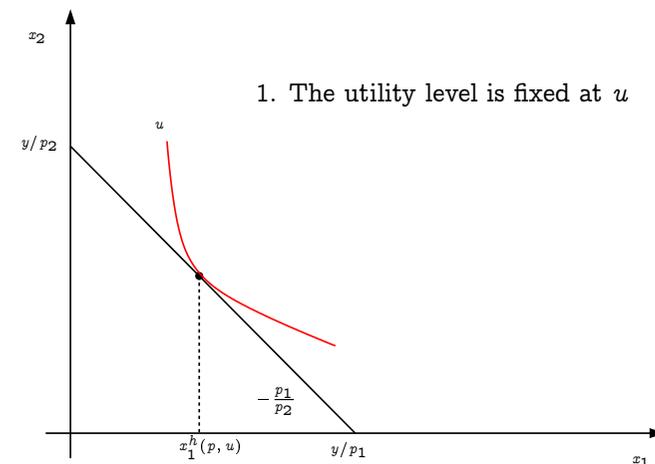
- since $px \geq 0$ and $p > 0$ then the set $\{e | e = px \text{ for some } x \text{ with } u(x) \geq u\}$ is closed and $e(p, u)$ is the smallest element of the set
- if $u(x)$ is quasiconcave then the “min” is unique and we can denote the solution to the minimization with a function $\hat{x}(p, u) \geq 0$ and as before

$$e(p, u) = p \hat{x}(p, u)$$

- how can we interpret $\hat{x}(p, u)$?

Hicksian demand function

Consider a standard CP, but fix the utility at level u .



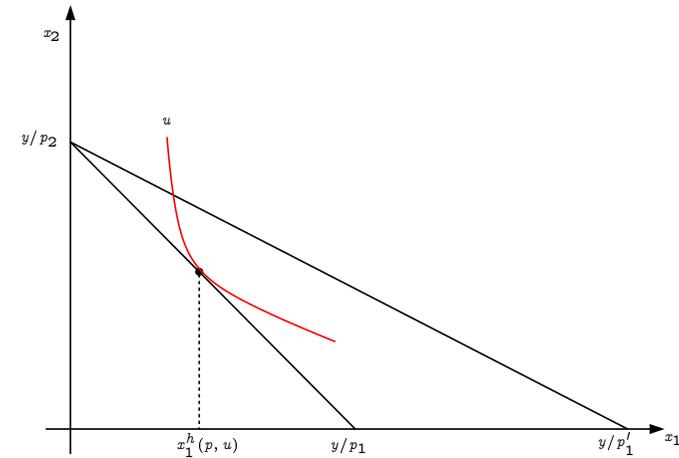
111 / 173

112 / 173

Hicksian demand function

Next, imagine to lower p_1 and to **penalize** the consumer reducing his income exactly of the amount needed to bring the consumer back to the previous utility level

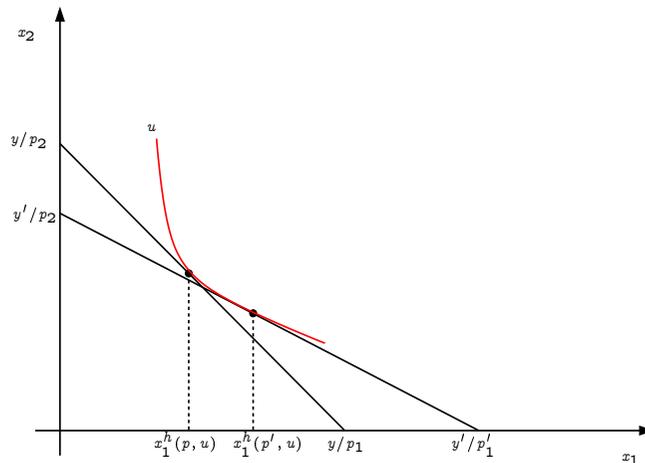
Hicksian demand function



113 / 173

114 / 173

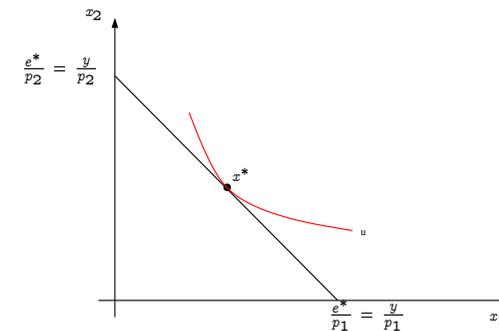
Hicksian demand function



$x^h(p', u)$ represents the consumption bundle chosen by the consumer at the new price level p' after being **compensated** to maintain constant his level of utility at u . $x^h(p, u)$ are called **Hicksian(compensated) demand functions**.

Expenditure-minimizing choice and Hicksian demand

The solution $\hat{x}(p, u)$ of the expenditure-minimization problem is the vector of the Hicksian demands $x^h(p, u)$.



EXPLANATION. Each of the hypothetical(compensated) budget constraint implies a level of expenditure exactly equal to minimum level necessary, at given prices, to achieve the utility level u

115 / 173

116 / 173

Theorem (Properties of $e(p, u)$)

If $u(x)$ is continuous and strictly increasing then $e(p, u)$ is

1. 0 when $u(x)$ takes on the lowest level of utility in \mathcal{U}
2. continuous on $\mathbb{R}_+ \times \mathcal{U}$
3. increasing and unbounded in $u \quad \forall p > 0$
4. increasing in p
5. homogeneous of degree 1 in p
6. concave in p

and if $u(x)$ is strictly quasi-concave

7. Shephard's Lemma:

$$\frac{\partial e(p^0, u^0)}{\partial p_i} = x_i^h(p^0, y^0)$$

117 / 173

Proof.(cont'ed).

5. It suffices to show that if x^* minimizes the expenditure at price p it also minimizes the expenditure at price tp . Suppose not, there exists x' such that

$$tpx' < tpx^* \Rightarrow px' < px \text{ contradiction!}$$

6. Concavity in p requires

$$te(p^0, u) + (1 - t)e(p^1, u) \leq e(tp^0 + (1 - t)p^1, u)$$

Now consider the following three expenditure minimizing combinations (p^1, x^1) , (p^2, x^2) and (p^3, x^3) where $p^3 = tp^1 + (1 - t)p^2$. Since x^3 is not necessarily minimizing the expenditure with p^1 and p^2 then

$$\begin{aligned} e(p^3, u) &= p^3 x^3 = tp^1 x^3 + (1 - t)p^2 x^3 \\ &\geq te(p^1, u) + (1 - t)e(p^2, u) \end{aligned}$$

119 / 173

Proof (sketch).

1. Since $u(x)$ is strictly increasing in x , its lowest value is $u(0)$. Then $px = 0$ and consequently $e(p, u(0)) = 0$.
2. It follows from the theorem of the maximum
3. Let's consider the Lagrangian $L = px - \lambda(-u(x) + u)$. If we assume that the constraint is binding and apply the Envelope theorem

$$\frac{\partial L}{\partial x_i} = 0 \Rightarrow p_i = -\lambda \frac{\partial u(x)}{\partial x_i} \Rightarrow \lambda < 0$$

and hence by the Envelope th.

$$\frac{d e(p, u)}{d u} = \frac{\partial L}{\partial u} = -\lambda > 0$$

4. cfr. property 7.

118 / 173

Proof.(cont'ed).

7. By the Envelope theorem

$$\frac{d e(p, u)}{d p_i} = \frac{\partial L}{\partial p_i} = x_i^* \equiv x_i^h$$

□

120 / 173

The relation between $v(p, y)$ and $e(p, u)$

Fixing p and y we get

$$u = v(p, y) \text{ then by definition } e(p, v(p, y)) \leq y$$

and analogously fixing p and u

$$y = e(p, u) \text{ then by definition } v(p, e(p, u)) \geq u .$$

Theorem

Let $v(p, y)$ and $e(p, u)$ be the indirect utility and the expenditure function for a consumer whose utility $u(x)$ is continuous and strictly increasing. Then $\forall p > 0, \forall y \geq 0$ and $\forall u \in \mathcal{U}$

$$i) e(p, v(p, y)) = y \quad ii) v(p, e(p, u)) = u$$

121 / 173

Proof.

i) Consider that $\mathcal{U} = [u(0), \bar{x}]$ with \bar{x} either finite or $+\infty$.

Suppose that i) does not hold

$$e(p, v(p, y)) < y \Rightarrow e(p, u) < y \text{ with } u = v(p, y)$$

$$\text{[by continuity of] } u(x) \quad e(p, u + \epsilon) < y \quad \epsilon > 0$$

$$\text{[hence there exists] } y_\epsilon : y_\epsilon = e(p, u + \epsilon) < y$$

$$\text{[since } v(p, y) \text{ is increasing in } y] \quad v(p, y) > v(p, y_\epsilon) \geq u + \epsilon$$

$$\Rightarrow u \geq u + \epsilon \text{ which contradicts } \epsilon > 0.$$

ii) Again by contradiction. Assume that

$$v(p, e(p, u)) > u \text{ with } e(p, u) = y$$

$$\text{[since } e() \text{ is increasing in } u] \quad y = e(p, u) > e(p, u(0)) = 0 \Rightarrow y > 0$$

$$\text{[by continuity of } v()] \quad \exists \epsilon > 0 : v(p, y - \epsilon) > u \Rightarrow e(p, u) \leq y - \epsilon$$

[which together with] $e(p, u) = y$ generate the contradiction

123 / 173

The relation between $v(p, y)$ and $e(p, u)$

Fixing p and y we get

$$u = v(p, y) \text{ then by definition } e(p, v(p, y)) \leq y$$

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$$i) e(p, v(p, y)) = y \quad ii) v(p, e(p, u)) = u$$

122 / 173

Duality of demand function

Theorem

Under the usual assumptions, $\forall p > 0, \forall y \geq 0$ and $\forall u \in \mathcal{U}$ we have

$$i) x_i(p, y) = x_i^h(p, v(p, y))$$

$$ii) x_i^h(p, u) = x_i(p, e(p, u)) .$$

This theorem states that the solution to the “Max u ” problem is also a solution of the “Min px ” problem and viceversa.

124 / 173

Proof.

Usual assumption guarantees the existence and uniqueness of the solution for the two problem $\Rightarrow v(p, y)$ and $e(p, u)$ are well defined. Assumes x^0 solves $\text{Max } u(x) \text{ s.t. } px - y \leq 0$, then we have

$$\begin{aligned}x^0 &= x(p, y^0) & u(x^0) &= u^0 \\v(p, y^0) &= u^0 & px^0 &= y^0 & \text{[by definition of } v()\text{]} \\e(p, u^0) &= e(p, v(p, y^0)) = y^0, & & \text{[by the previous theorem]}\end{aligned}$$

which states that x^0 is a solution of the $\text{Min } px \text{ st } u(x) \geq u^0$ and hence

$$x^0 = x^h(p, u^0) = x^h(p, v(p, y^0))$$

Analogously for ii).



125 / 173

Income and prices

People do (should) care about real commodities.

- **RELATIVE PRICE:** number of units of some other commodity that must be foregone to acquire 1 unit of a given good

$$\frac{p_i}{p_j} = \frac{\text{€}/\text{unit } i}{\text{€}/\text{unit } j} = \frac{\text{unit } j}{\text{unit } i} \text{ relative price of good } i$$

- **REAL INCOME:** maximum number of units of some commodity the consumer could acquire if he spends his entire money income

$$\frac{y}{p_i} = \frac{\text{€}}{\text{€}/\text{unit } i} = \text{unit } i$$

127 / 173

Introduction

preferences
utility
choice

Consumer Theory

preference relations
utility function

Quick Maths Refresh

The Consumer Problem

indirect utility function
expenditure function
the Slutsky equation

Revealed Preferences

Uncertainty

126 / 173

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128 / 173

Homogeneity and budget balancedness(BB)

Theorem

Under usual conditions the demand function $x_i(p, y) \forall i$ is homogeneous of degree zero in all prices and income and satisfies the budget balancedness

$$px(p, y) = y \quad \forall(p, y)$$

Proof.

From the properties of the indirect utility we know

$$v(p, y) = v(tp, ty) \Leftrightarrow u(x(p, y)) = u(x(tp, ty))$$

and since the budget sets with (p, y) and (tp, ty) are equal then both $x(p, y)$ and $x(tp, ty)$ were feasible when the other was chosen.

129 / 173

Proof. (cont'ed).

Finally the previous equality and from strict quasi-concavity of $u(x)$ we get

$$x(p, y) = x(tp, ty)$$

The second part follows directly from the fact that $u(x)$ is strictly increasing.

□

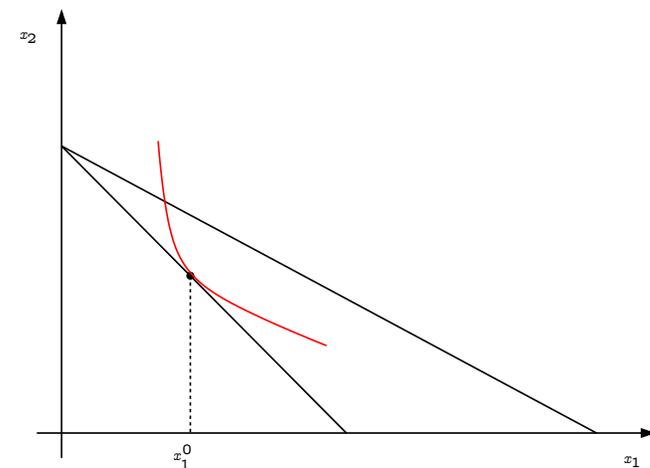
130 / 173

Effects of a price change

What response should we expect in the demanded quantity of a commodity when its price changes? Suppose we observe a reduction in the price of x_1

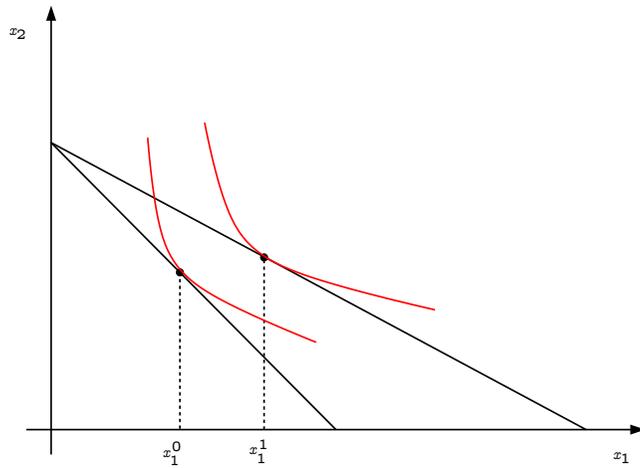
131 / 173

Effects of a price change



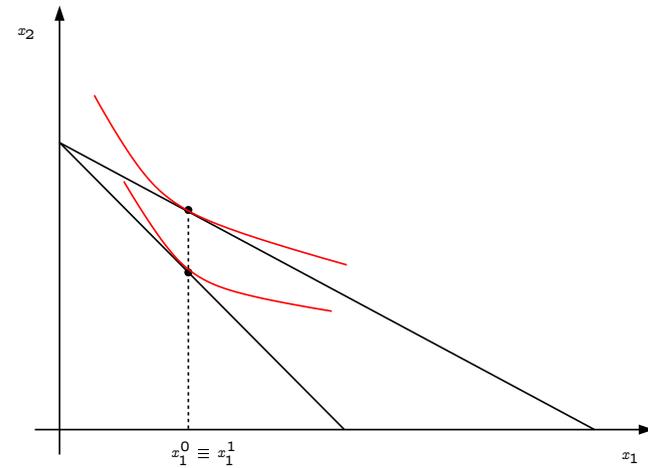
132 / 173

Effects of a price change



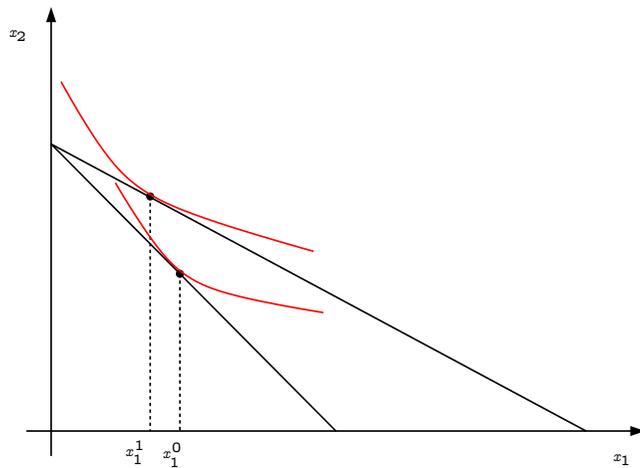
133 / 173

Effects of a price change



134 / 173

Effects of a price change



135 / 173

Effects of a price change

Intuitively when a price changes there are two reasons why we should expect a change in the corresponding demand:

- the good whose price declined becomes relatively cheaper compared to the other good; since they are all desirable we expect the consumer to substitute cheaper good with the more expensive one [SUBSTITUTION EFFECT]
- when the price of any good declines the consumer total command over all goods is increased allowing him to increase his consumption of all good in the way he sees fit [INCOME EFFECT]

136 / 173

Effects of a price change

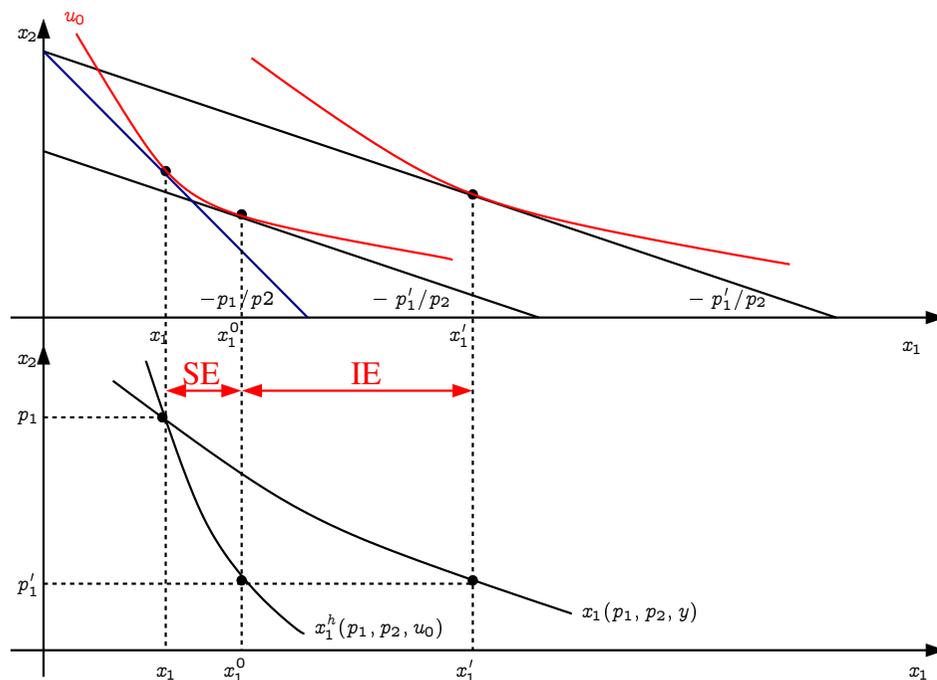
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- the good whose price declined becomes relatively cheaper compared to the other good; since they are all desirable we expect the consumer to substitute cheaper good with the more expensive one [SUBSTITUTION EFFECT]
- when the price of any good declines the consumer total command over all goods is increased allowing him to increase his consumption of all good in the way he sees fit [INCOME EFFECT]

Suppose for example to observe a reduction of the price of x_1 from p_1 to p'_1

137 / 173

138 / 173



Theorem (Slutsky equation)

Let $x(p, y)$ be the marshallian demand system of a consumer. Let u^* be the utility level the consumer achieves at price p and income y . Then

$$\frac{\partial x_i(p, y)}{\partial p_j} = \frac{\partial x_i^h(p, u^*)}{\partial p_j} - x_j(p, y) \frac{\partial x_i(p, y)}{\partial y} \quad \forall i, j$$

139 / 173

140 / 173

Theorem (Slustky equation)

Let $x(p, y)$ be the marshallian demand system of a consumer. Let u^* be the utility level the consumer achieves at price p and income y . Then

$$\underbrace{\frac{\partial x_i(p, y)}{\partial p_j}}_{TE} = \underbrace{\frac{\partial x_i^h(p, u^*)}{\partial p_j}}_{SE} - \underbrace{x_j(p, y) \frac{\partial x_i(p, y)}{\partial y}}_{IE} \quad \forall i, j$$

Proof. (cont'ed).

Now by the Shephard's Lemma and by considering that $e(p, u^*) = e(p, v(p, y)) = y$ we get

$$\frac{\partial x_i(p, y)}{\partial p_j} = \frac{\partial x_i^h(p, u^*)}{\partial p_j} - x_j(p, y) \frac{\partial x_i(p, y)}{\partial y}$$

□

REMARK. Note that the Hicksian demands are unobservable. What can we know about Hicksian demands if we cannot even directly see them? Quite a bit.

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Proof.

Consider

$$\begin{aligned} x_i^h(p, u^*) &= x_i(p, e(p, u^*)) \\ \frac{\partial x_i^h(p, u^*)}{\partial p_j} &= \frac{\partial x_i(p, e(p, u^*))}{\partial p_j} + \frac{\partial x_i(p, e(p, u^*))}{\partial y} \frac{\partial e(p, u^*)}{\partial p_j} \\ &\quad \text{[by total differentiation]} \end{aligned}$$

Proof. (cont'ed).

Now by the Shephard's Lemma and by considering that $e(p, u^*) = e(p, v(p, y)) = y$ we get

$$\frac{\partial x_i(p, y)}{\partial p_j} = \frac{\partial x_i^h(p, u^*)}{\partial p_j} - x_j(p, y) \frac{\partial x_i(p, y)}{\partial y}$$

□

REMARK. Note that the Hicksian demands are unobservable. What can we know about Hicksian demands if we cannot even directly see them? Quite a bit.

Theorem (Negative own-substitution terms)

Let $x_i^h(p, u)$ be the Hicksian demand for good i . Then

$$\frac{\partial x_i^h(p, u)}{\partial p_i} \leq 0 \quad \forall i$$

Proof.

From the Shephard's lemma we have

$$\begin{aligned} \frac{\partial e(p, u)}{\partial p_i} &= x_i^h(p, u) \\ \frac{\partial^2 e(p, u)}{\partial p_i^2} &= \frac{\partial x_i^h(p, u)}{\partial p_i} \leq 0 \quad \text{[since } e(p, u) \text{ is concave in } p\text{]} \end{aligned}$$

□

IMPLICATION: Law of Demand for normal good.

145 / 173

Theorem (Negative semi-definite substitution matrix)

Let $x_i^h(p, u)$ be the Hicksian demand for good i and let

$$\sigma(p, u) \equiv \begin{bmatrix} \frac{\partial x_1^h(p, u)}{\partial p_1} & \cdots & \frac{\partial x_1^h(p, u)}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n^h(p, u)}{\partial p_1} & \cdots & \frac{\partial x_n^h(p, u)}{\partial p_n} \end{bmatrix}$$

be the substitution matrix. Then $\sigma(p, u)$ is negative semidefinite.

Proof.

For each element of $\sigma(p, u)$ it holds that

$$\frac{\partial x_i^h(p, u)}{\partial p_j} = \frac{\partial^2 e(p, u)}{\partial p_j \partial p_i}$$

The theorem then follows from the concavity of $e(p, u)$.

147 / 173

Theorem (Symmetric substitution term)

Let $x_i^h(p, u)$ be the Hicksian demand for good i . Then

$$\frac{\partial x_i^h(p, u)}{\partial p_j} = \frac{\partial x_j^h(p, u)}{\partial p_i} \quad \forall i, j$$

Proof.

From the Shephard's lemma we have

$$\begin{aligned} \frac{\partial e(p, u)}{\partial p_i} &= x_i^h(p, u) \Rightarrow \frac{\partial x_i^h(p, u)}{\partial p_j} = \frac{\partial^2 e(p, u)}{\partial p_j \partial p_i} \\ \frac{\partial^2 e(p, u)}{\partial p_j \partial p_i} &= \frac{\partial^2 e(p, u)}{\partial p_j \partial p_i} \quad \text{[by Young's theorem]} \\ \frac{\partial x_i^h(p, u)}{\partial p_j} &= \frac{\partial x_j^h(p, u)}{\partial p_i} \quad \forall i, j \end{aligned}$$

□

146 / 173

Theorem (Slutsky matrix)

Let $x(p, y)$ be the Marshallian demand system. Define the Slutsky matrix as

$$\begin{bmatrix} \frac{\partial x_1(p, y)}{\partial p_1} + x_1(p, y) \frac{\partial x_1(p, y)}{\partial y} & \cdots & \frac{\partial x_1(p, y)}{\partial p_n} + x_1(p, y) \frac{\partial x_1(p, y)}{\partial y} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n(p, y)}{\partial p_1} + x_n(p, y) \frac{\partial x_n(p, y)}{\partial y} & \cdots & \frac{\partial x_n(p, y)}{\partial p_n} + x_n(p, y) \frac{\partial x_n(p, y)}{\partial y} \end{bmatrix}$$

Then the Slutsky matrix $s(p, y)$ is symmetric and negative semi-definite.

Proof.

Consider the Slutsky equation

$$\frac{\partial x_i^h(p, u^*)}{\partial p_j} = \frac{\partial x_i(p, y)}{\partial p_j} + x_j(p, y) \frac{\partial x_i(p, y)}{\partial y}$$

then $\sigma(p, u) = s(p, y)$ and hence $s(p, y)$ is symmetric and negative semi-definite.

148 / 173

Testing the theory I

If a marshallian demand system is to be viewed as to a **price-taking utility-maximizing consumer** then

1. demands must be homogeneous
2. demands must satisfy BB
3. the associated Slutsky matrix must be symmetric and negative semi-definite

These requirements provide a set of restrictions on allowable values for the parameters in any empirically estimated marshallian demand system.

Elasticities

We define

$$\eta_i \equiv \frac{\partial x_i(p, y)}{\partial y} \frac{y}{x_i(p, y)} \quad [\text{income elasticity}]$$

$$\epsilon_{ij} \equiv \frac{\partial x_i(p, y)}{\partial p_j} \frac{p_j}{x_i(p, y)} \quad [\text{cross-price elasticity}]$$

$$s_i \equiv \frac{p_i x_i(p, y)}{y} \quad s_i \geq 0 \quad \sum_{i=1}^n s_i = 1$$

Theorem

Aggregations The following aggregating relations hold

$$\text{i) } \sum_{i=1}^n \eta_i s_i = 1 \quad [\text{Engel Aggregation}]$$

$$\text{ii) } \sum_{i=1}^n \epsilon_{ij} s_i = -s_j \quad [\text{Cournot Aggregation}]$$

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149 / 173

150 / 173

Proof.

i) Differentiating the BB wrt y

$$1 = \sum_{i=1}^n p_i \frac{\partial x_i(p, y)}{\partial y} = \sum_{i=1}^n p_i \frac{x_i}{y} \frac{\partial x_i(p, y)}{\partial y} \frac{y}{x_i} = \sum_{i=1}^n \eta_i s_i$$

ii) Differentiating the BB wrt p_j

$$0 = \sum_{i \neq j} p_i \frac{\partial x_i(p, y)}{\partial p_j} + x_j(p, y) + p_j \frac{\partial x_j(p, y)}{\partial p_j}$$

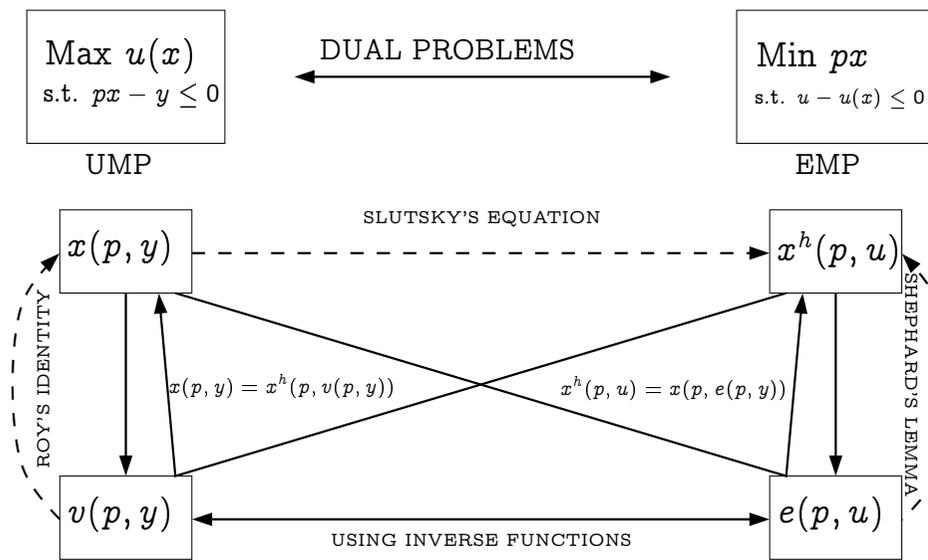
$$0 = \sum_{i=1}^n p_i \frac{\partial x_i(p, y)}{\partial p_j} + x_j(p, y)$$

$$\sum_{i=1}^n p_i \frac{x_i}{y} \frac{\partial x_i(p, y)}{\partial p_j} \frac{p_j}{x_i} = -x_j(p, y) \frac{p_j}{y}$$

$$\sum_{i=1}^n \epsilon_{ij} s_i = -s_j$$

151 / 173

152 / 173



RI: $x_i(p, y) = -\frac{\partial v(p, y) / \partial p_i}{\partial v(p, y) / \partial y}$ SL: $\frac{\partial e(p, u)}{\partial p_i} = x_i^h(p, u)$

SE: $\frac{\partial x_i(p, y)}{\partial p_j} = \frac{\partial x_i^h(p, u)}{\partial p_j} - x_j(p, y) \frac{\partial x_i(p, y)}{\partial y}$

Introduction

- preferences
- utility
- choice

Consumer Theory

- preference relations
- utility function

Quick Maths Refresh

The Consumer Problem

- indirect utility function
- expenditure function
- the Slutsky equation

Revealed Preferences

Uncertainty

Intro

The Theory of Demand has been derived by assuming consumers have preferences satisfying some axioms and deducing both demand functions and their properties.

Samuelson(1947): Why not start and finish with observable behaviors?

INTUITION. If the consumer demands one consumption bundle instead of another affordable one then we may say that the first bundle is revealed preferred to the second.

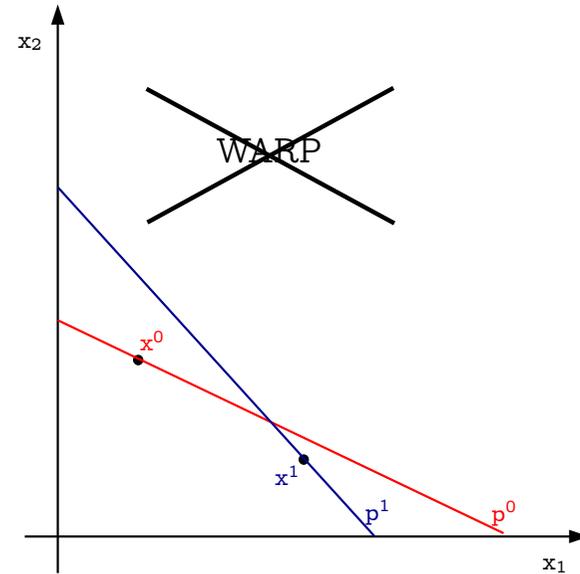
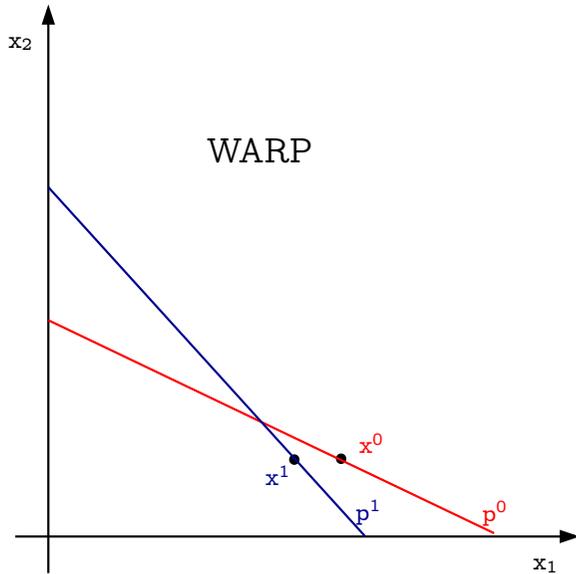
Weak Axiom of Revealed Preferences - WARP

Definition (WARP)

A consumer's choice behavior satisfies WARP if $\forall x^0, x^1$ with x^0 chosen at p^0 and x^1 chosen at p^1

$$p^0 x^1 \leq p^0 x^0 \Rightarrow p^1 x^0 > p^1 x^1$$

WARP holds if whenever x^0 is revealed preferred to x^1 , x^1 is never revealed preferred to x^0 .



Demand function and WARP

Does the Marshallian demand satisfies the WARP?

Under usual assumptions we know that the solution to the UMP is unique. Suppose that at p^0 and p^1 our consumer chooses

$$x^0 \text{ and } x^1 \text{ and that } p^1 x^1 \leq p^0 x^0 .$$

Then since x^1 is affordable and not chosen $u(x^0) > u(x^1)$. At p^1 on the contrary, x^0 is not chosen so it must be $p^1 x^0 > p^1 x^1$. Hence

$$p^1 x^1 \leq p^0 x^0 \Rightarrow p^1 x^0 > p^1 x^1 \text{ [WARP]}$$

WARP and the choice function

Assume:

- consumer's behavior satisfies WARP
- $x(p, y)$ is his **choice function** (not a demand function!)
- BB: $\forall p > 0 \quad px(p, y) = y$

Under these assumptions the **choice function $x(p, y)$ is homogeneous of degree 0 in (p, y)**

Slutsky compensated demand

Proof.

Suppose at p^0 the consumer chooses x^0 and hence $p^0 x^0 = y^0$. Analogously at p^1 he chooses x^1 and hence $p^1 x^1 = y^1$. Assume $p^1 = tp^0$ and consequently $y^1 = ty^0$. From BB it follows that

$$p^1 x^1 = p^0 x^0 \Rightarrow \begin{cases} tp^0 x^1 = tp^0 x^0 \\ p^1 x^1 = p^1 x^0 \end{cases} \Rightarrow \begin{cases} p^0 x^1 = p^0 x^0 \\ p^1 x^1 = p^1 x^0 \end{cases} .$$

IF $x^0 \neq x^1$ then WARP implies $p^1 x^1 < p^1 x^0$, **contradiction**. Hence, it must be $x^1 = x^0$ but this means that $x(p, y)$ is homogeneous of degree zero.

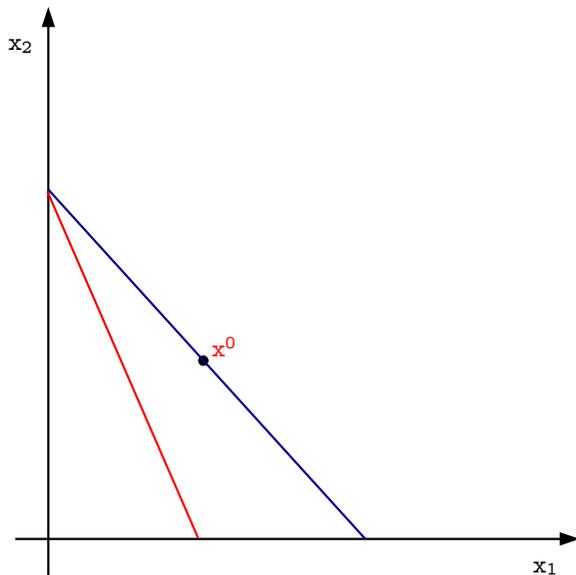
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The Slutsky compensation when prices vary arbitrarily is obtained by changing the income so that the consumer can just afford the initial bundle.

161 / 173

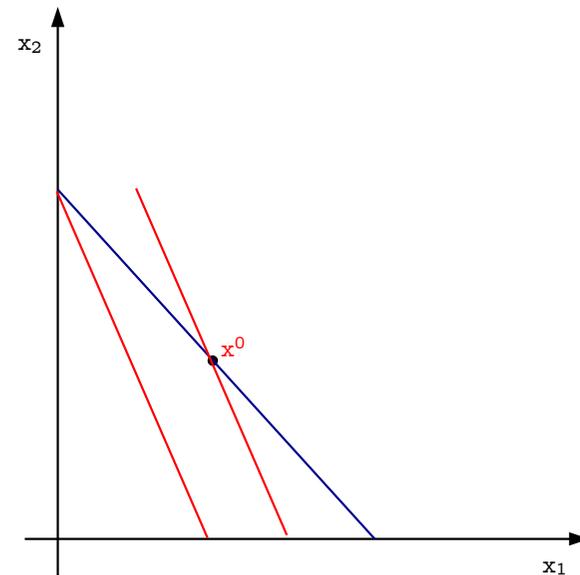
162 / 173

Slutsky compensated demand



163 / 173

Slutsky compensated demand



164 / 173

Slutsky compensated demand

The Slutsky compensation when prices vary arbitrarily is obtained by changing the income so that the consumer can just afford the initial bundle.

Consequently, under these circumstances, at price p the consumer's income will be px^0 and his the function, describing his choice behavior, $x(p, px^0)$,

Under these assumptions the **Slutsky matrix associated with the choice function $x(p, y)$ is negative semidefinite.**

Proof. (Cont'ed).

Since the last holds $\forall p^1$, it holds also for $p^1 = p^0 + tz \quad \forall t > 0$ and $\forall x \in \mathbb{R}^n$ (negative and positive). This implies

$$(p^0 + tz - p^0)x^0 \geq (p^0 + tz - p^0)x(p^0 + tz, (p^0 + tz)x^0)$$

$$zx^0 \geq zx(p^0 + tz, (p^0 + tz)x^0) \quad .$$

Since $p^0 > 0$, for fixed z we may choose $\bar{t} > 0$ small enough so that $p^0 + tz > 0 \quad \forall t \in [0, \bar{t})$. Then

$$\text{IF } t = 0 \Rightarrow zx^0 = zx(p^0, p^0x^0)$$

$$\text{THEN } f(t) \equiv zx(p^0 + tz, (p^0 + tz)x^0)$$

is maximized on $[0, \bar{t})$ at $t = 0$ and hence $f'(t)|_{t=0} \leq 0$. Let's take the derivative of $f(t)$.

Proof.

Consider the following situation

$$p^0 > 0 \quad y^0 > 0 \qquad p^0 > 0 \quad y^0 > 0$$

$$x^0 = x(p^0, p^0x^0) \qquad x^1 = x(p^1, p^1x^0)$$

$$[\text{WARP}][*] \quad p^0x^0 \leq p^0x^1 \text{ since } \begin{cases} x^0 = x^1 \Rightarrow p^0x^0 = p^0x^1 \\ x^0 \neq x^1 \Rightarrow p^0x^0 < p^0x^1 \end{cases}$$

$$[\text{BB}][*] \quad p^1x^0 = p^1x(p^1, p^1x^0)$$

$$[*] - [**] \quad (p^1x^0 - p^0x^0) \leq p^1x(p^1, p^1x^0) - p^0x^1$$

$$(p^1 - p^0)x^0 \geq (p^1 - p^0)x(p^1, p^1x^0) \quad \forall p^1$$

165 / 173

166 / 173

Two goods example

$$f(t) \equiv zx(p^0 + tz, (p^0 + tz)x^0)$$

$$\frac{df(t)}{dt} = z_1 \left[\frac{\partial x_1}{\partial p_1} \frac{dp_1}{dt} + \frac{\partial x_1}{\partial p_2} \frac{dp_2}{dt} + \frac{\partial x_1}{\partial y} \frac{dy}{dt} \right] +$$

$$\left[\frac{\partial x_2}{\partial p_1} \frac{dp_1}{dt} + \frac{\partial x_2}{\partial p_2} \frac{dp_2}{dt} + \frac{\partial x_2}{\partial y} \frac{dy}{dt} \right]$$

$$\frac{df(t)}{dt} = z_1 \left[\frac{\partial x_1}{\partial p_1} z_1 + \frac{\partial x_1}{\partial p_2} z_2 + \frac{\partial x_1}{\partial y} (z_1x_1 + z_2x_2) \right] +$$

$$\left[\frac{\partial x_2}{\partial p_1} z_1 + \frac{\partial x_2}{\partial p_2} z_2 + \frac{\partial x_2}{\partial y} (z_1x_1 + z_2x_2) \right]$$

$$= z_1 \left[\sum_{j=1}^2 \frac{\partial x_1}{\partial p_j} z_j + \frac{\partial x_1}{\partial y} z_j x_j \right] + z_2 \left[\sum_{j=1}^2 \frac{\partial x_2}{\partial p_j} z_j + \frac{\partial x_2}{\partial y} z_j x_j \right]$$

$$= \sum_{i=1}^2 z_i \sum_{j=1}^2 \frac{\partial x_i}{\partial p_j} z_j + \frac{\partial x_i}{\partial y} x_j z_j \quad .$$

167 / 173

168 / 173

Proof. (Cont'ed).

Hence in the n-goods case

$$\frac{df(t)}{dt} = \sum_{i=1}^n z_i \sum_{j=1}^n \frac{\partial x_i}{\partial p_j} z_j + \frac{\partial x_i}{\partial y} x_j z_j \leq 0 \quad .$$

Since $z \in \mathbb{R}^n$ is arbitrary this means the matrix whose ij element is

$$\frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial y} x_j$$

is negative semidefinite. But this matrix is precisely **the Slutsky matrix associated with the choice function $x(p, y)$.**

□

169 / 173

We have just proven the following

Theorem

If a choice function $x(p, y)$ satisfies WARP and BB then it is homogeneous of degree zero in (p, y) and possesses a Slutsky matrix negative semidefinite.

What is missing to complete the recovery of a full fledged demand function? To assure symmetry WARP is not enough.

Definition (SARP)

SARP is satisfied IF, for any sequence of distinct consumption bundles x^0, x^1, \dots, x^k where x^0 is revealed preferred to x^1 , x^1 to x^2 ... x^{k-1} to x^k , THEN it's not the case that x^k is revealed preferred to x^0 .

170 / 173

Introduction

- preferences
- utility
- choice

Consumer Theory

- preference relations
- utility function

Quick Maths Refresh

The Consumer Problem

- indirect utility function
- expenditure function
- the Slutsky equation

Revealed Preferences

Uncertainty

171 / 173

172 / 173

